

The geometry of Lie algebroids and its applications to optimal control

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Abstract

The paper presents the geometry of Lie algebroids and its applications to optimal control. The first part deals with the theory of Lie algebroids, connections on Lie algebroids and dynamical systems defined on Lie algebroids (mainly Lagrangian and Hamiltonian systems). In the second part we use the framework of Lie algebroids in the study of distributional systems (drift less control affine systems) with holonomic or nonholonomic distributions.

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PREFACE

The present paper is devoted to Lie algebroids geometry and its applications to optimal control and variational calculus. The framework of the differential geometry is very useful in modelling and understanding of a large class of natural phenomena. The Lie geometric methods are applied successfully in differential equations, optimal control theory or theoretical physics. In the most of cases the study is starting with a variational problem formulated for a regular Lagrangian (see [1]), on the tangent bundle TM over the manifold M and very often the whole set of problems is transferred on the dual space T^*M , endowed with a Hamiltonian function, via Legendre transformation. The case of a non-regular Lagrangians is also studied. The problem in this case is that the proposed Lagrangian formalism yields a singular Lagrangian description, which makes the Legendre transform ill-defined and thus no straightforward Hamiltonian formulation can be related. One of the motivations for the present work is the study of Lagrangian systems subjected to external constraints (holonomic or nonholonomic). These systems have a wide application in many different areas as optimal control theory, mathematical economics or sub-Riemannian geometry.

In the last years the investigations have led to a geometric framework which is covering these phenomena. It is precisely the underlying structure of a Lie algebroid on the phase space which allows a unified treatment. This idea was first introduced by A. Weinstein [125, 23] in order to define a Lagrangian formalism which is very useful for the various types of such systems.

The concept of Lie algebroids have been introduced into differential geometry since the early 1950, and also can be found in physics and algebra, under a wide variety of names. However, the fundamental concept has been introduced in sixties by J. Pradines [112] in relation with Lie groupoids. For every Lie groupoid there exists an associated Lie algebroid, like as for every Lie group there exists an associated Lie algebra. A Lie algebroid [73, 75] over a smooth manifold M is a real vector bundle (E, π, M) with a Lie algebra structure on its space of sections, and an application σ , named anchor, which induces a Lie algebra homomorphism from sections of E to vector fields on M . It is convenient to think a Lie algebroid as a substituent for the tangent bundle of M , an element e of E as a generalized velocity, and the actual velocity v on TM is obtained when applying the anchor to e , i.e., $\sigma(e) = v$.

The basic example of Lie algebroid over the manifold M is the tangent bundle TM itself, with the identity mapping as anchor. Every integrable distribution of TM is a Lie algebroid with the inclusion as anchor and induced Lie bracket, and every Lie algebra is a Lie algebroid over one point. An important Lie algebroid is the cotangent bundle of a Poisson manifold

[60]. Being related to many areas of geometry, as connections theory [75, 40, 24, 31, 49, 76, 111, 95] cohomology [73, 75] foliations and pseudogroups, symplectic and Poisson geometry [66, 124, 120, 39, 121, 61, 33, 36, 97, 99, 107] the Lie algebroids are today the object of extensive studies. More precisely, Lie algebroids have applications in mechanical systems and optimal control theory [30, 77, 78, 49, 29, 44, 4, 92, 96, 98, 16] (distributional systems) and are a natural framework in which one can be developed the theory of differential operators (exterior derivative and Lie derivative) and differential equations.

In his papers [73, 75] K. Mackenzie has been achieved a unitary study of Lie groupoids and algebroids and together with P. Higgins [46] have introduced the notion of prolongation of a Lie algebroid over a smooth map, useful in the study of induced vector bundle by the Lie algebroid structure. Using the geometry of Lie algebroids, A. Weinstein [125] shows that is possible to give a common description of the most interesting classical mechanical systems. He developed a generalized theory of Lagrangian mechanics and obtained the equations of motions, using the Poisson structure on the dual of a Lie algebroid and Legendre transformation associated with a regular Lagrangian. In the last years the problems raised by A. Weinstein have been investigated by many authors. Thus, E. Martinez [69, 70, 72] obtained the same Euler-Lagrange equations using the symplectic formalism for Lagrangian and Hamiltonian, similarly with the J. Klein formalism [57] for the classical Lagrangian mechanics.

In the classical version of the tangent bundle ($E = TM$) the Klein's method is based on the vector bundle structure of TM and the existence of a vector-valued 1-form. Such a form does not exist for a general Lie algebroid [65] because of different dimensions of the horizontal and vertical distributions, and so Klein's approach is not applied directly. To overcome this difficulty, E. Martinez, M de Leon, J. C. Marero [69, 64] have proposed a modified version, in which the bundles tangent to E and E^* are replaced by the prolongations $\mathcal{T}E$ and $\mathcal{T}E^*$ (in sense Higgins and Mackenzie [46]). The nonholonomic Lagrangian systems and Hamiltonian mechanics on Lie algebroids are studied by a group of E. Martinez [64]. The first step in studying the mechanical control systems on Lie algebroids seems to be done by J. Cortes and E. Martinez [30], which also approached the problem of accessibility and controllability. A framework for nonholonomic systems, using a subbundle of a Lie algebroids is proposed by T. Mestdag and B. Langerock [77]. A start in the study of some problems of control affine systems and sub-Riemannian geometry, using the framework of Lie algebroids is due to D. Hrimiuc and L. Popescu [49, 96, 98].

Control theory is splitting in two major branches: the first is the control theory of problems described by partial differential equations where the objective functionals are mostly quadratic forms, and the second is the control theory of problems described by the parameter dependent ordinary differ-

ential equations. In this last case it is more frequent to deal with non-linear systems and non-quadratic objective functional. The mathematical models from the optimal control theory cover also the economic growth in both open and closed economies, exploitation of (non-) renewable resources, pollution control, behavior of firms or differential games [38, 115, 116].

The geometric methods in the control theory have been applied by many authors (see [18, 56, 15, 71]). One of the most important issues in the geometric approach is the analysis of the solution to the optimal control problem as provided by Pontryagin's Maximum Principle; that is, the curve $c(t) = (x(t), u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ satisfying the Hamilton equations, together with a maximization condition for the Hamiltonian with respect to the control variables $u(t)$.

In the paper [71] E. Martinez presents the Pontryagin Maximum Principle on Lie algebroids using the prolongation (in sense of Higgins and Mackenzie [46]) of the Lie algebroid over the vector bundle projection of a dual bundle. In this paper we study some distributional systems with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid.

"In spite of that, the control theory can be considered part of the general theory of differential equations, the problems that inspires it and some of the results obtained so far, have configured a theory with a strong and definite personality, that is already offering interesting returns to its ancestors. For instance, the geometrization of non-linear affine-input control theory problems by introducing Lie-geometrical methods into its analysis, started already by R. Brockett [18], is inspiring classical Riemannian geometry and creating what is called today sub-Riemannian geometry" [118, 85, 14, 2, 22, 9, 10, 11, 12, 13].

If M is a smooth n -dimensional manifold then a sub-Riemannian structure on M is a pair (D, g) where D is a distribution of rank m and g is a Riemannian metric on D . A piecewise smooth curve on M is called horizontal if its tangent vectors are in D . The length of a horizontal curve c is defined by

$$L(c) = \int_I \sqrt{g(\dot{c}(t))} dt, \quad (1)$$

where g is a Riemannian metric on D . The distance between two points a and b is $d(a, b) = \inf L(c)$, where the infimum is taken over all horizontal curves connecting a to b . The distance is assumed to be infinite if there is no horizontal curve that connects these two points. If locally, the distribution D of rank m is generated by X_i , $i = \overline{1, m}$ a sub-Riemannian structure on M is locally given by a control system

$$\dot{x} = \sum_{i=1}^m u_i(t) X_i(x), \quad (2)$$

of constant rank m , with the controls $u(\cdot)$. The controlled paths are obtained by integrating the system (2) and are the geodesics in the framework of sub-Riemannian geometry. If D is assumed to be bracket generating, *i.e.* sections of D and iterated brackets span the entire tangent space TM , by a well-known theorem of Chow [25] the system (2) is controllable, that is for any two points a and b , there exists a horizontal curve which connects these points (M is assumed to be connected).

The concept of sub-Riemannian geometry can be extended to a more general setting, [49, 27, 28] by replacing the Riemannian metric with a positive homogeneous one. For the theory of optimal control this extension is equivalent to the change of the quadratic cost of a control affine system with a positive homogeneous cost. Also, the case of distribution D with non-constant rank is generating interesting examples (Grushin case [37, 49]).

The case when the distribution D generated by vector fields X_i , $i = \overline{1, m}$ is integrable is also studied. In this case the distribution determines a foliation on M and two points can be joined if and only if belongs to the same leaf. In order to find the optimal trajectory of the system one uses the Pontryagin Maximum Principle at the level of Lie algebroids, built different in the case of holonomic or nonholonomic distributions.

* * *

The paper is organized in two parts. The first part entitled *The geometry of Lie algebroids* contains seven chapters. In the first chapter some preliminaries concerning geometrical structures on the total space of a vector bundle are presented [82]. We focus on the notions of nonlinear connection and covariant derivative. In the next chapter we present the notion of *Lie algebroid* including the cohomology and structure equations [73]. The notion of prolongation of a Lie algebroid over the vector bundle projection is studied in the chapter three. The Ehresmann nonlinear connection $\mathcal{N} = -\mathcal{L}_S J$ with the coefficients given by

$$\mathcal{N}_\alpha^\beta = \frac{1}{2} \left(-\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L_{\alpha\varepsilon}^\beta \right),$$

is investigated and the relations with the Ehresmann connections on tangent bundles TE and TM are pointed out. In the chapter four we introduce the notion of dynamical covariant derivative and metric nonlinear connection at the level of the Lie algebroid \mathcal{TE} . The Lagrangian formalism on Lie algebroids yields a canonical semispray [70]

$$\mathcal{S}^\varepsilon = g^{\varepsilon\beta} \left(\sigma_\beta^i \frac{\partial L}{\partial x^i} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L_{\beta\alpha}^\theta y^\alpha \frac{\partial L}{\partial y^\theta} \right),$$

and a canonical Ehresman connection, which is a metric nonlinear connection. We also have the Lagrange equations on Lie algebroids given by [125]

$$\frac{dx^i}{dt} = \sigma_\alpha^i y^\alpha, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) = \sigma_\alpha^i \frac{\partial L}{\partial x^i} - L_{\alpha\beta}^\theta y^\beta \frac{\partial L}{\partial y^\theta}.$$

In the case of positive homogeneous Lagrangian (Finsler function) we find a canonical Ehresmann connection which depends only on Finsler function and the structure functions of the Lie algebroid.

In the chapter five we deal with the prolongation of a Lie algebroid over the vector bundle projections of a dual bundle. We introduce the notions of dual adapted tangent structure \mathcal{J} and \mathcal{J} -regular sections. These structures induce a canonical nonlinear connection $\mathcal{N} = -\mathcal{L}_\rho \mathcal{J}$ with the coefficients given by [50]

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2} \left(t_{\alpha\gamma} \frac{\partial \rho_\beta}{\partial \mu_\gamma} - \sigma_\alpha^i t_{\gamma\beta} \frac{\partial \xi^\gamma}{\partial q^i} - \rho(t_{\alpha\beta}) + \xi^\gamma t_{\lambda\beta} L_{\gamma\alpha}^\lambda \right).$$

In the case of Hamiltonian formalism these coefficients become [103]

$$\begin{aligned} \mathcal{N}_{\alpha\beta} = & \frac{1}{2} (\sigma_\gamma^i \{g_{\alpha\beta}, \mathcal{H}\} - \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_\varepsilon} (\sigma_\beta^i g_{\alpha\varepsilon} + \sigma_\alpha^i g_{\beta\varepsilon}) + \\ & + \mu_\gamma L_{\varepsilon\kappa}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\varepsilon} \frac{\partial g_{\alpha\beta}}{\partial \mu_\kappa} + \mu_\gamma L_{\alpha\beta}^\gamma + \frac{\partial \mathcal{H}}{\partial \mu_\delta} (g_{\alpha\varepsilon} L_{\delta\beta}^\varepsilon + g_{\beta\varepsilon} L_{\delta\alpha}^\varepsilon)), \end{aligned}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. The corresponding Hamilton equations on Lie algebroid are given by [125, 64]

$$\frac{dq^i}{dt} = \sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial q^i} - \mu_\gamma L_{\alpha\beta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\beta}.$$

In the chapter six we introduce the notion of dynamical covariant derivative and metric nonlinear connection at the level of a Lie algebroid \mathcal{TE}^* . We prove that the canonical nonlinear connection induces by a regular Hamiltonian is a unique metric and symmetric nonlinear connection. In the chapter seven we investigate some aspects of the Lie algebroids geometry endowed with a Poisson structures, the so-called Poisson-Lie algebroids.

Author's papers [49, 50, 91, 92, 94, 95, 96, 97, 99, 100, 103, 104, 105, 106, 107, 109] are used in writing this part.

The purpose of the second part entitled *Optimal Control* is to study the drift less control affine systems (distributional systems) with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid in the case of constant rank of distribution.

We prove that the framework of Lie algebroids is better than cotangent bundle in order to solve some problems of drift less control affine systems. In the first chapter the known results on the optimal control systems are recalled by geometric viewpoint. In the next chapter the distributional systems are presented and the relation between the Hamiltonians on E^* and T^*M is given by

$$H(p) = \mathcal{H}(\mu), \quad \mu = \sigma^*(p), \quad p \in T_x^*M, \quad \mu \in E_x^*.$$

We investigate the cases of holonomic and nonholonomic distributions with constant rank. In the holonomic case, we will consider the Lie algebroid being just the distribution whereas in the nonholonomic case (i.e., strong bracket generating distribution) the Lie algebroid is the tangent bundle with the basis given by vectors of distribution completed by the first Lie brackets. Also, the case of distribution D with non-constant rank is studied in the last two sections of the chapter and some interesting examples are given. In the last chapter we present the intrinsic relation between the distributional systems and sub-Riemannian geometry. Thus, the optimal trajectory of our distributional systems are the geodesics in the framework of sub-Riemannian geometry. We investigate two classical cases: Grusin plan and Heisenberg group, but equipped with positive homogeneous costs (Randers metric). We are using the Pontryagin Maximum Principle at the level of Lie algebroids, in the case of Heisenberg group and show that this idea is very useful in order to solve a large class of distributional systems. Author's papers [50, 96, 98, 101, 102, 106, 108] are used in writing this part.

In my opinion, the paper is useful to a large class of readers: graduate students, mathematicians and to everybody else interested in the subject

of differential geometry, differential equations or optimal control. I want to address my thanks to all authors mentioned in this paper and to everybody else I forgot to mention, without any intention, in the Bibliography.

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1 THE GEOMETRY OF LIE ALGEBROIDS

The purpose of this first part is to study the geometry of a Lie algebroid and its prolongations over the vector bundles projections. A Lie algebroid [73, 75] over a smooth manifold M is a real vector bundle (E, π, M) with a Lie algebra structure on its space of sections, and an application σ , named the anchor, which induces a Lie algebra homomorphism from the sections of E to vector fields on M . For this reason, in the first chapter we present some results on the geometry of the total space of a vector bundle, including nonlinear connections and covariant derivatives. In the next chapter we give only the relevant formulas for Lie algebroid cohomology we shall need later, and refer the reader to the monograph [73] for further details.

The chapter three deals with the prolongation $\mathcal{T}E$ of a Lie algebroid over the vector bundle projection. We introduce the Ehresmann nonlinear connection on the Lie algebroid $\mathcal{T}E$ and study its properties [95, 92]. We show that the vertical part of the Lie brackets of horizontal sections from the basis represents the components of the curvature tensor of the nonlinear connection. We study the related connections and show that a connection on the tangent bundle TE induces a connection on the Lie algebroid $\mathcal{T}E$. We introduce an almost complex structure on Lie algebroids and prove that its integrability is characterized by zero torsion and curvature property of the connection. We present the notion of dynamical covariant derivative at the level of a Lie algebroid and show that the metric compatibility of the semispray and associated nonlinear connection gives the one of the so called Helmholtz conditions of the inverse problem of Lagrangian Mechanics. In the homogeneous case a canonical nonlinear connection associated to a Finsler function is determined. We study the linear connections on $\mathcal{T}E$ and determine the torsion and curvature.

In the chapter four we study the dynamical covariant derivative and metric nonlinear connection on $\mathcal{T}E$ [104]. We introduce the dynamical covariant derivative as a tensor derivation and study the compatibility conditions with a pseudo-Riemannian metric. In the case of SODE connection we find the expression of Jacobi endomorphism and its relation with curvature tensor. We prove that the canonical nonlinear connection induced by a regular Lagrangian is a unique connection which is metric and compatible with symplectic structure. Also the invariant form of the Helmholtz conditions on Lie algebroids are given (see also [35]).

The chapter five deals with the prolongation $\mathcal{T}E^*$ of a Lie algebroid over the vector bundle projection of a dual bundle. We study the properties of the

connections on $\mathcal{T}E^*$ [50, 94, 96] and introduce the notions of adapted almost tangent structure, almost complex structure and characterize the integrability conditions in terms of torsion and curvature of the connection. We prove that every \mathcal{J} -regular section (in particular, any regular Hamiltonian on E^*) determines a canonical Ehresmann connection on the Lie algebroid $\mathcal{T}E^*$. We introduce some generalizations of the Hamilton sections, as a mechanical structures and semi-Hamiltonian sections and study their properties. In the last part of this chapter, using the diffeomorphism from $\mathcal{T}E^*$ and $\mathcal{T}E$ [64] induced by a regular Hamiltonian, we can transfer many geometrical results between these spaces. Thus, a semispray on $\mathcal{T}E$ is transformed into a semi-Hamiltonian section on $\mathcal{T}E^*$ if and only if the nonlinear connection on $\mathcal{T}E$ determined by semispray is just the canonical nonlinear connection induced by regular Lagrangian, via Legendre transformation.

In the chapter six we study the dynamical covariant derivative and metric nonlinear connection on $\mathcal{T}E^*$ [109]. Using the notion of \mathcal{J} -regular section we introduce the dynamical covariant derivative as a tensor derivation. In the case of nonlinear connection induced by a \mathcal{J} -regular section we find the expression of Jacobi endomorphism and its relation with curvature tensor. Finally, we prove that the canonical nonlinear connection induced by a regular Hamiltonian is the unique metric and symmetric nonlinear connection.

In the chapter seven we investigate some aspects of the Lie algebroids geometry endowed with a Poisson structures [97, 99, 107], which generalize the Poisson manifolds. We recall the Cartan calculus and the Schouten-Nijenhuis bracket at the level of Lie algebroids and introduce the Poisson structure on Lie algebroids. We study the properties of linear contravariant connection and its tensors of torsion and curvature. In the last part of this section we find a Poisson connection which depends only on the Poisson bivector and structural functions of Lie algebroid, which generalize some results of Fernandes from [39]. Also the geodesic equations are given. We study the properties of the complete lift of a Poisson bivector on $\mathcal{T}E$ and introduce the notion of horizontal lift. The compatibility conditions of these bivectors are investigated. Finally, the compatibility conditions between the canonical Poisson structure and the horizontal lift on $\mathcal{T}E^*$ are given.

1.1 The geometry of the total space of a vector bundle

1.1.1 Connections on vector bundles

The connections theory is an important topic of the differential geometry with important applications in Differential Equations or Optimal Control. In this section we shall present only the notion of Ehresmann nonlinear connection and induced geometrical structures. Also, the covariant derivative is described. For more details and complete proofs we refer to the monographs [59, 63, 79, 82].

Let us consider the n -dimensional differentiable manifold M and a vector bundle (E, π, M) over M , with type fibre $F = \mathbb{R}^m$. The structure of the vector bundle is given by a vectorial atlas $\{(U_i, \psi_i, \mathbb{R}^m)\}_{i \in I}$ such that

- 1) $(U_i)_{i \in I}$ is an open covering of the manifold M .
- 2) The mappings $\psi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^m$ are bijective and satisfy the relation

$$\pi(\psi_i^{-1}(x, f)) = x, \quad x \in M, \quad f \in \mathbb{R}^m.$$

- 3) For every pair $(i, j) \in I \times I$, such that $U_i \cap U_j \neq \emptyset$ there exists a smooth mapping $g_{ij} : U_i \cap U_j \rightarrow GL(m, \mathbb{R})$ with $\Psi_{i,x}^{-1} = \Psi_{j,x}^{-1} \circ g_{ji}(x)$ for every $x \in U_i \cap U_j$ where $\Psi_{i,x}^{-1}$ is the restriction of ψ_i^{-1} to $\{x\} \times \mathbb{R}^m$.

Let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas on the manifold M such that U_i belongs to the maps domain into vectorial atlas $\{(U_i, \psi_i, \mathbb{R}^m)\}_{i \in I}$. We obtain that $\{(\pi^{-1}(U_i), h_i)\}_{i \in I}$ with

$$h_i : \pi^{-1}(U_i) \rightarrow \mathbb{R}^n \times \mathbb{R}^m, \quad h_i(u) = (\varphi_i(\pi(u)), \psi_{i,\pi(u)}(u)),$$

is a differentiable atlas on the manifold E . If (U_j, φ_j) is another local chart on M with $U_i \cap U_j \neq \emptyset$ and $(U_j, \psi_j, \mathbb{R}^m)$ is a bundle chart, then

$$(h_j \circ h_i^{-1})(x, y) = ((\varphi_j \circ \varphi_i)(x), g_{ji}(\varphi_i(x))y), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

We will denote by $M_a^{a'}(x)$ the entries of the matrix associated with the linear application $g_{ji}(\varphi_i(x))$. For $x \in U \subset M$ we take $\varphi_i(x) = (x^i) \in \mathbb{R}^n$, $i = \overline{1, n}$ and considering $\varphi_j(x) = (x^{i'})$ then $\varphi_j \circ \varphi_i^{-1}$ has the form

$$x^{i'} = x^{i'}(x^1, \dots, x^n), \quad \text{rank} \left(\frac{\partial x^{i'}}{\partial x^i} \right) = n.$$

A tangent vector X_x at the point $x \in M$ will be locally represented by the pair (x^i, y^i) , where (y^i) is given by $X_x = y^i \frac{\partial}{\partial x^i}$. Therefore, the transformations of coordinates on differentiable manifold TM have the form

$$x^{i'} = x^{i'}(x^1, \dots, x^n), \quad \text{rank} \left(\frac{\partial x^{i'}}{\partial x^i} \right) = n, \quad y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i. \quad (1.1.1)$$

A point $u \in E$ on the total space of a vector bundle (E, π, M) is locally represented by the pair $(x^i, y^a) \in \mathbb{R}^n \times \mathbb{R}^m$. The transformations of coordinates $(x^i, y^a) \rightarrow (x^{i'}, y^{a'})$ on E are of the form [82]

$$\begin{aligned} x^{i'} &= x^{i'}(x^1, \dots, x^n), \quad \text{rank} \left(\frac{\partial x^{i'}}{\partial x^i} \right) = n, \\ y^{a'} &= M_a^{a'}(x) y^a, \quad \text{rank}(M_a^{a'}(x)) = m. \end{aligned} \tag{1.1.2}$$

For the vector bundle (E, π, M) we consider $\pi_* : TE \rightarrow TM$ the tangent application of π . The application π_* is a π -morphism of vector bundles between tangent bundles (TE, π_E, E) and (TM, π_M, M) . The kernel of this π -morphism is a subbundle of vector bundle (TE, π_E, E) , denoted $(VE, \pi_V, E) = \text{Ker } \pi_*$, which will be called the *vertical subbundle*. The total space is $VE = \cup V_u$, where $V_u = \text{Ker } \pi_*$, $u \in E$.

A tangent vector X_u at the point $u \in E$ has the local representation (x^i, y^a, X^i, A^a) , where the coefficients $(X^i) \in \mathbb{R}^n$ and $(A^a) \in \mathbb{R}^m$ are defined by the relation $X_u = X^i \frac{\partial}{\partial x^i} + A^a \frac{\partial}{\partial y^a}$. The tangent application π_* is locally represented by $\pi_*(x, y, X, A) = (x, X)$. Consequently, the local fibres of vector bundle (TE, π_*, TM) are isomorph with $\{x\} \times \mathbb{R}^m \times \{X\} \times \mathbb{R}^m \simeq \mathbb{R}^{2m}$. Because $\pi_* \left(\frac{\partial}{\partial y^a} \right) = 0$, it results that the functions $\left\{ \frac{\partial}{\partial y^a} \right\}, a = \overline{1, m}$ determine a local basis of the vertical distribution $\{u \rightarrow V_u \mid u \in E\}$, which means that V_u is integrable. The elements of the vertical subbundle have the form $(x, y, 0, A)$, that is the fibres of vertical subbundle VE are locally isomorph with \mathbb{R}^m . Let $\pi^* TM$ be the induced vector bundle of tangent bundle over the application $\pi : E \rightarrow M$ and $\pi! : TE \rightarrow \pi^* TM$ given by

$$\pi!(X_u) = (u, \pi_*(X_u)).$$

This application is a morphism of vector bundles, and follows that the application $\pi!$ is a surjection and

$$\text{Ker } \pi! = \text{Ker } \pi_* = VE.$$

Therefore, it results that the following sequence of vector bundles over E is exact

$$0 \rightarrow VE \xrightarrow{i} TE \xrightarrow{\pi!} \pi^* TM \rightarrow 0. \tag{1.1.3}$$

where $i : VE \rightarrow TE$ is the inclusion map.

We can present now a definition of the Ehresmann connection, called usually nonlinear connection.

Definition 1.1.1 *The nonlinear connection in the vector bundle (E, π, M) is a splitting on the left of the exact sequences (1.1.3).*

It results that a nonlinear connection in (E, π, M) is a morphism of vector bundle $N : TE \rightarrow VE$ such that $N \circ i = \text{Id}|_{VE}$. The kernel of the

morphism N is a vector subbundle of the bundle (TE, π_E, E) , which will be called *horizontal subbundle* and will be denoted (HE, π_H, E) . Therefore, the vector bundle (TE, π_E, E) is the Whitney sum of the horizontal and vertical subbundles. Thus we have the following characterization of the nonlinear connection.

Proposition 1.1.1 *A nonlinear connection in (E, π, M) is determined by the existence of the vector subbundle (HE, π_H, E) of the tangent bundle over E , (TE, π_E, E) such that $TE = HE \oplus VE$.*

The restriction $\pi!|_{HE}$ of the application $\pi!$ to the horizontal subbundle HE is an isomorphism of vector bundles. The component $\pi_* : HE \rightarrow TM$ of the application $\pi!|_{HE}$ is a π -morphism and its restriction to the fibres is an isomorphism. Therefore, for any vector field X on M , there exists a horizontal vector field on E , such that $\pi_*(X^h) = X$. The vector field X^h is called the *horizontal lift* of the vector field X . The horizontal lift has a local representation

$$\left(\frac{\partial}{\partial x^i}\right)^h = \frac{\delta}{\delta x^i}, \quad i = \overline{1, n},$$

thus, we determine a local basis $\{\frac{\delta}{\delta x^i}\}$ of $H_u E$. This vector field can be represented in the form $\frac{\delta}{\delta x^i} = A_i^j \frac{\partial}{\partial x^j} + B_i^a \frac{\partial}{\partial y^a}$, but the condition $\pi_*\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial x^i}$ implies $A_j^i = \delta_j^i$ and the fact that $\frac{\delta}{\delta x^i}$ are the kernel of the mapping N gives $B_i^a = -N_i^a$. It results

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \quad (1.1.4)$$

We obtain a new basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\right)$ of the tangent bundle $T_u E$ which is called the *Berwald basis* associated to the nonlinear connection N .

If we denote N_i^a , $i = \overline{1, n}$, $a = \overline{1, m}$ the coefficients of a nonlinear connection, then it results [82]:

Proposition 1.1.2 *By a change of the local coordinates (1.1.2) on the vector bundle (E, π, M) , the local coefficients N_i^a of a nonlinear connection N change as follows*

$$N_{i'}^{a'} \frac{\partial x^{i'}}{\partial x^i} = M_a^{a'} N_i^a - \frac{\partial M_a^{a'}}{\partial x^i} y^a. \quad (1.1.5)$$

A nonlinear connection in the vector bundle $(E \setminus \{0\}, \pi, M)$ is said to be homogeneous (respective linear) if the coefficients $N_i^a(x, y)$ are homogeneous (respective linear) with respect to the second argument.

We define a morphism $v : \mathcal{X}(E) \rightarrow \mathcal{X}(E)$ such that $v(X) = -N(X)$ if $X \in \Gamma(VE)$ and $v(X) = 0$, for $X \in \Gamma(HE)$. It follows that the morphism $v : \mathcal{X}(E) \rightarrow \mathcal{X}(E)$ has the properties:

$$1^\circ v(\mathcal{X}(E)) \subset \Gamma(VE),$$

$$2^\circ \{v(X) = X\} \Leftrightarrow X \in \Gamma(VE),$$

and it determines a nonlinear connection in (E, π, M) since $Kerv = \Gamma(HE)$.

It results that $v^2 = v$, *i.e.* v is a projector, which is called the *vertical projector* of the nonlinear connection N . Analogously, a nonlinear connection in the vector bundle (E, π, M) is characterized by the morphism $h : \mathcal{X}(E) \rightarrow \mathcal{X}(E)$ with the properties $h^2 = h$, $Ker h = \Gamma(VE)$, and h is called the *horizontal projector* of the nonlinear connection N . It results that $h + v = Id$. In [82] one proves:

Proposition 1.1.3 *A nonlinear connection in a vector bundle (E, π, M) is characterized by an almost product structure P on E whose distribution of eigensubspaces which correspond to the eigenvalue -1 coincides to the vertical distribution.*

From the previous consideration it follows that

$$P = 2h - Id = Id - 2v = h - v, P^2 = Id.$$

The existence of the nonlinear connection in (E, π, M) leads to the decomposition

$$\mathcal{X}(E) = \Gamma(HE) \oplus \Gamma(VE) \Rightarrow X = hX + vX, \forall X \in \mathcal{X}(E).$$

1.1.2 Covariant derivative

We consider a local basis $\{s_a\}$ of the sections of the bundle $\pi^{-1}(U) \rightarrow U$, $U \subset M$ and it results $\rho = \rho^a(x)s_a$, $\rho \in \Gamma(E)$. The nonlinear connection N induces a covariant derivative for the sections in E defined as follows

$$D_X \rho = X^i \left(\frac{\partial \rho^a}{\partial x^i} + N_i^a(x, \rho(x)) X^i \right) s_a,$$

where

$$X = X^i \frac{\partial}{\partial x^i} \in \Gamma(TM).$$

This covariant derivative has the properties:

$$1^\circ D_{X+Y} = D_X + D_Y,$$

$$2^\circ D_{fX} \rho = f D_X \rho, \quad X, Y \in \Gamma(TM).$$

If N is homogeneous one also has

$$3^\circ D_X(f\rho) = X(f)\rho + f D_X \rho, \quad \rho \in \Gamma(E), \quad f \in \mathcal{F}(M).$$

If N is linear, then the covariant derivative satisfies $1^\circ, 2^\circ, 3^\circ$ and

$$4^\circ D_X(\rho_1 + \rho_2) = D_X \rho_1 + D_X \rho_2, \quad \rho_1, \rho_2 \in \Gamma(E).$$

Let $c: [a, b] \rightarrow M$, $t \rightarrow c(t)$ be a smooth curve on the manifold M and $\dot{c}: [a, b] \rightarrow TM$ the tangent field along the curve c . Setting $D_{\dot{c}} \rho = \frac{D\rho}{dt}$ we say that the section ρ in (E, π, M) is parallel along c if $\frac{D\rho}{dt} = 0$ and it results:

Proposition 1.1.4 *A local section ρ of the bundle $\pi^{-1}(U) \rightarrow U$, $U \subset M$ is parallel along the curve c with $c([a, b]) \subset U$ if and only if*

$$\frac{d\rho^a}{dt} + N_i^a(x, \rho(x)) \frac{dx^i}{dt} = 0, \quad \rho = \rho^a(x)s_a, \quad x \in U.$$

The decomposition $T_u E = H_u E \oplus V_u E$ in the vector bundle (E, π, M) leads to the decomposition $(T_u E)^* = (V_u E)^\perp \oplus (H_u E)^\perp$, for $u \in E$, where $(V_u E)^\perp$ denotes the subspace of 1-forms on $T_u E$ which vanish on horizontal vectors, and $(H_u E)^\perp$ denotes the subspace of 1-forms on $T_u E$ which vanish on vertical vectors. Therefore, we have

$$\omega = h\omega + v\omega, \quad \forall \omega \in \wedge^1(E),$$

where $h\omega(X) = \omega(hX)$ and $v\omega(X) = \omega(vX)$. The dual Berwald basis is $(dx^i, \delta y^a)$, where

$$\delta y^a = dy^a + N_i^a dx^i. \quad (1.1.6)$$

Proposition 1.1.5 *The Berwald basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\right)$ and its dual $(dx^i, \delta y^a)$ transform under a change of coordinates (1.1.2) as follows*

$$\frac{\delta}{\delta x^i} = \frac{\partial x^{i'}}{\partial x^i} \frac{\delta}{\delta x^{i'}}, \quad \frac{\partial}{\partial y^a} = M_a^{a'} \frac{\partial}{\partial y^{a'}},$$

$$dx^i = \frac{\partial x^i}{\partial x^{i'}} dx^{i'}, \quad \delta y^a = M_{a'}^a \delta y^{a'}.$$

The Lie brackets of the vector fields of Berwald basis are given by

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = R_{ji}^a \frac{\partial}{\partial y^a}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a} \right] = \frac{\partial N_i^b}{\partial y^a} \frac{\partial}{\partial y^b}, \quad \left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right] = 0, \quad (1.1.7)$$

where

$$R_{jk}^a = \frac{\delta N_j^a}{\delta x^k} - \frac{\delta N_k^a}{\delta x^j}. \quad (1.1.8)$$

In the following we deal with the curvature of a nonlinear connection N .

Definition 1.1.2 *The curvature of a nonlinear connection N is given by*

$$\Omega = -\mathbf{N}_v, \quad (1.1.9)$$

where v is the vertical projector induced by N and \mathbf{N}_v is the Nijenhuis tensor associated to v ,

$$\mathbf{N}_v(X, Y) = [vX, vY] - v[vX, Y] - v[X, vY] + v^2[X, Y], \quad X, Y \in \mathcal{X}(E).$$

In local coordinates, we set

$$\Omega = \frac{1}{2} \Omega_{ij}^a (dx^i \wedge dx^j) \otimes \frac{\partial}{\partial y^a},$$

and using (1.1.9), we obtain

$$\Omega(hX, hY) = -v[hX, hY], \quad \Omega(hX, vY) = \Omega(vX, vY) = 0, \quad (1.1.10)$$

and then

$$\Omega_{ij}^a = -R_{ij}^a = \frac{\delta N_j^a}{\delta x^i} - \frac{\delta N_i^a}{\delta x^j}. \quad (1.1.11)$$

The equality $v = Id - h$ leads to $\mathbf{N}_v = \mathbf{N}_h$, and we obtain $\Omega = -\mathbf{N}_h$.

Proposition 1.1.6 *The following equation holds*

$$[X^h, Y^h] = [X, Y]^h - \Omega(X^h, Y^h).$$

Using (1.1.10) follows a characterization of the integrability of the horizontal distribution.

Theorem 1.1.1 *The horizontal distribution of a nonlinear connection N is integrable if and only if the curvature vanishes.*

See [20, 82] for the particular case of the tangent bundle.

1.2 Lie algebroids

1.2.1 Cohomology

Let M be a real, C^∞ -differentiable, n -dimensional manifold and (TM, π_M, M) its tangent bundle.

Definition 1.2.1 *A Lie algebroid over a manifold M is a triple $(E, [\cdot, \cdot]_E, \sigma)$, where (E, π, M) is a vector bundle of rank m over M , which satisfies the following conditions:*

- a) $C^\infty(M)$ -module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot, \cdot]_E$.
- b) $\sigma : E \rightarrow TM$ is a bundle map (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from the Lie algebra of sections $(\Gamma(E), [\cdot, \cdot]_E)$ to the Lie algebra of vector fields $(\chi(M), [\cdot, \cdot])$ satisfying the Leibniz rule

$$[s_1, fs_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2, \quad \forall s_1, s_2 \in \Gamma(E), \quad f \in C^\infty(M). \quad (1.2.1)$$

From the above definition it results:

1° $[\cdot, \cdot]_E$ is a \mathbb{R} -bilinear operation,

2° $[\cdot, \cdot]_E$ is skew-symmetric, i.e.

$$[s_1, s_2]_E = -[s_2, s_1]_E, \quad \forall s_1, s_2 \in \Gamma(E),$$

3° $[\cdot, \cdot]_E$ verifies the Jacobi identity

$$[s_1, [s_2, s_3]_E]_E + [s_2, [s_3, s_1]_E]_E + [s_3, [s_1, s_2]_E]_E = 0, \quad (1.2.2)$$

and σ being a Lie algebra homomorphism, means that

$$\sigma[s_1, s_2]_E = [\sigma(s_1), \sigma(s_2)]. \quad (1.2.3)$$

The existence of a Lie bracket on the space of sections of a Lie algebroid leads to a calculus on its sections analogous to the usual Cartan calculus on differential forms. In this paragraph we give only the relevant formulas for Lie algebroid cohomology we shall need later, and refer the reader to the monograph [73] for further details.

If f is a function on M , then $df(x) \in E_x^*$ is given by $\langle df(x), a \rangle = \sigma(a)f$, for $\forall a \in E_x$. For $\omega \in \bigwedge^k(E^*)$ the exterior derivative $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned}
d^E \omega(s_1, \dots, s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \sigma(s_i) \omega(s_1, \dots, \hat{s}_i, \dots, s_{k+1}) + \\
&+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([s_i, s_j]_E, s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{k+1}).
\end{aligned} \tag{1.2.4}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and the hat over an argument means the absence of the argument. It results that

$$(d^E)^2 = 0, \tag{1.2.5}$$

$$d^E(\omega_1 \wedge \omega_2) = d^E \omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d^E \omega_2, \tag{1.2.6}$$

The cohomology associated with d^E is called the *Lie algebroid cohomology* of E and is denoted by $H^\bullet(E)$. Also, for $\xi \in \Gamma(E)$ one can define the *Lie derivative* with respect to ξ by

$$\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi, \tag{1.2.7}$$

where i_ξ is the contraction with ξ .

1.2.2 Structure equations on Lie algebroids

If we take the local coordinates (x^i) on an open $U \subset M$, a local basis $\{s_\alpha\}$ of the sections of the bundle $\pi^{-1}(U) \rightarrow U$ generates local coordinates (x^i, y^α) on E . The local functions $\sigma_\alpha^i(x)$, $L_{\alpha\beta}^\gamma(x)$ on M given by

$$\sigma(s_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta]_E = L_{\alpha\beta}^\gamma s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m}, \tag{1.2.8}$$

are called the *structure functions of the Lie algebroid*, and satisfy the *structure equations* on Lie algebroids

$$\sum_{(\alpha, \beta, \gamma)} \left(\sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\delta}{\partial x^i} + L_{\alpha\eta}^\delta L_{\beta\gamma}^\eta \right) = 0, \tag{1.2.9}$$

$$\sigma_\alpha^j \frac{\partial \sigma_\beta^i}{\partial x^j} - \sigma_\beta^j \frac{\partial \sigma_\alpha^i}{\partial x^j} = \sigma_\gamma^i L_{\alpha\beta}^\gamma. \tag{1.2.10}$$

Locally, if $f \in C^\infty(M)$ then $d^E f = \frac{\partial f}{\partial x^i} \sigma_\alpha^i s^\alpha$, where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$ and if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d^E \theta = \left(\sigma_\alpha^i \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2} \theta_\gamma L_{\alpha\beta}^\gamma \right) s^\alpha \wedge s^\beta. \tag{1.2.11}$$

Particularly, we get

$$d^E x^i = \sigma_\alpha^i s^\alpha, \quad d^E s^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha s^\beta \wedge s^\gamma. \tag{1.2.12}$$

Under a change of coordinates

$$\begin{cases} x^{i'} = x^{i'}(x^i), & i, i' = \overline{1, n} \text{ on } M, \\ y^{\alpha'} = A_{\alpha}^{\alpha'}(x^i) y^{\alpha}, & \alpha, \alpha' = \overline{1, m} \text{ on } E, \end{cases} \quad (1.2.13)$$

corresponding to a new base $s_{\alpha} = A_{\alpha}^{\alpha'} s_{\alpha'}$, then the transformation rules of the structure functions are

$$\sigma_{\alpha'}^{i'} A_{\alpha}^{\alpha'} = \frac{\partial x^{i'}}{\partial x^i} \sigma_{\alpha}^i, \quad (1.2.14)$$

$$L_{\alpha\beta}^{\gamma} A_{\gamma}^{\gamma'} = L_{\alpha'\beta'}^{\gamma'} A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} + \sigma_{\alpha}^i \frac{\partial A_{\beta}^{\gamma'}}{\partial x^i} - \sigma_{\beta}^i \frac{\partial A_{\alpha}^{\gamma'}}{\partial x^i}. \quad (1.2.15)$$

Some examples of Lie algebroids which will be used in this paper (see [75] for more examples and details)

Example 1.2.1 *The tangent bundle $E = TM$ itself, with identity mapping as anchor. Then $H^{\bullet}(E) = H_{de \text{ Rham}}^{\bullet}(M)$ is the de Rham cohomology. With respect to the usual coordinates (x, \dot{x}) , the structure functions are $L_{jk}^i = 0$, $\sigma_j^i = \delta_j^i$, but if we were to change to another basis for the vector fields, the structure functions would become nonzero.*

Example 1.2.2 *Any integrable subbundle of TM is a Lie algebroid with the inclusion as anchor and the induced bracket. So, $E = T\mathcal{F} \subset TM$, an involutive distribution associated with a regular foliation \mathcal{F} , where one gets the tangential cohomology denoted $H_{\mathcal{F}+}^{\bullet}(M)$.*

Example 1.2.3 *The cotangent bundle of a Poisson manifold $E = T^*M$ with (M, Π) Poisson manifold. These carry a bracket characterized by the rule $\{df, dg\} = d\{f, g\}$ and the anchor is the map $\Pi^{\#} : T^*M \rightarrow TM$ associated to the Poisson bivector field Π . We obtain the Poisson cohomology denoted $H_{\Pi}^{\bullet}(M)$ [120].*

1.3 The prolongation of a Lie algebroid over the vector bundle projection

Let (E, π, M) be a vector bundle. For the projection $\pi : E \rightarrow M$ we can construct the prolongation of E (see [46, 69, 70, 64]). The associated vector bundle is $(\mathcal{T}E, \pi_2, E)$ where

$$\mathcal{T}E = \cup \mathcal{T}_w E, \quad w \in E,$$

$$\mathcal{T}_w E = \{(u_x, v_w) \in E_x \times T_w E \mid \sigma(u_x) = T_w \pi(v_w), \quad \pi(w) = x \in M\},$$

and the projection $\pi_2(u_x, v_w) = \pi_E(v_w) = w$, where $\pi_E : TE \rightarrow E$ is the tangent projection.

We also have the canonical projection $\pi_1 : \mathcal{T}E \rightarrow E$ given by $\pi_1(u, v) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E \rightarrow TE$, $\sigma^1(u, v) = v$ will be the anchor of a new Lie algebroid over the manifold E . An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_1 .

We will denote $(V\mathcal{T}E, \pi_2|_{V\mathcal{T}E}, E)$ the vertical bundle of $(\mathcal{T}E, \pi_2, E)$ and

$$\sigma^1|_{V\mathcal{T}E} : V\mathcal{T}E \rightarrow VTE,$$

is an isomorphism. If $f \in C^\infty(M)$ we will denote by f^c and f^v the *complete* and *vertical lift* to E of f defined by

$$f^c(u) = \sigma(u)(f), \quad f^v(u) = f(\pi(u)), \quad u \in E.$$

For $s \in \Gamma(E)$ we can consider the *vertical lift* of s given by

$$s^v(u) = s(\pi(u))_u^v,$$

for $u \in E$, where

$$\begin{smallmatrix} v \\ u \end{smallmatrix} : E_{\pi(u)} \rightarrow T_u(E_{\pi(u)})$$

is the canonical isomorphism.

There exists a unique vector field s^c on E , the *complete lift* of s satisfying the two following conditions:

- i) s^c is π -projectable on $\sigma(s)$,
- ii) $s^c(\hat{\alpha}) = \widehat{\mathcal{L}_s \alpha}$,

for all $\alpha \in \Gamma(E^*)$, where $\hat{\alpha}(u) = \alpha(\pi(u))(u)$, $u \in E$ (see [41, 42]).

Considering the prolongation $\mathcal{T}E$ of E over the projection π , we may introduce the *vertical lift* s^v and the *complete lift* s^c of a section $s \in \Gamma(E)$ as the sections of $\mathcal{T}E \rightarrow E$ given by (see [70])

$$s^v(u) = (0, s^v(u)), \quad s^c(u) = (s(\pi(u)), s^c(u)), \quad u \in E.$$

Another two canonical objects on $\mathcal{T}E$ are the *Euler section* C and the *almost tangent structure* (vertical endomorphism) J .

Definition 1.3.1 The Euler section C is the section of $\mathcal{T}E \rightarrow E$ defined by

$$C(u) = (0, u_u^v), \quad \forall u \in E.$$

Definition 1.3.2 The vertical endomorphism J is the section of the bundle $(\mathcal{T}E) \oplus (\mathcal{T}E)^* \rightarrow E$ characterized by

$$J(s^v) = 0, \quad J(s^c) = s^v, \quad s \in \Gamma(E).$$

The vertical endomorphism satisfies

$$J^2 = 0, \quad \text{Im} J = \ker J = V\mathcal{T}E, \quad [C, J]_{\mathcal{T}E} = -J.$$

Definition 1.3.3 A section \mathcal{S} of $\mathcal{T}E \rightarrow E$ is called semispray (or second order differential equation -SODE) on E if

$$J(\mathcal{S}) = C. \quad (1.3.1)$$

The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$, where [70]

$$\mathcal{X}_\alpha(u) = \left(s_\alpha(\pi(u)), \sigma_\alpha^i \frac{\partial}{\partial x^i} \Big|_u \right), \quad \mathcal{V}_\alpha(u) = \left(0, \frac{\partial}{\partial y^\alpha} \Big|_u \right), \quad (1.3.2)$$

and $(\partial/\partial x^i, \partial/\partial y^\alpha)$ is the local basis on TE . The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha}, \quad (1.3.3)$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_{\mathcal{T}E} = L_{\alpha\beta}^\gamma \mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0. \quad (1.3.4)$$

If V is a section of $\mathcal{T}E$, then in terms of basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$ it is

$$V = Z^\alpha \mathcal{X}_\alpha + V^\alpha \mathcal{V}_\alpha,$$

and the vector field $\sigma^1(V) \in \chi(E)$ has the expression

$$\sigma^1(V) = \sigma_\alpha^i Z^\alpha \frac{\partial}{\partial x^i} + V^\alpha \frac{\partial}{\partial y^\alpha}.$$

The vertical lift of a section $\rho = \rho^\alpha s_\alpha$ and the corresponding vector field are

$$\rho^v = \rho^\alpha \mathcal{V}_\alpha,$$

respectively

$$\sigma^1(\rho^v) = \rho^\alpha \frac{\partial}{\partial y^\alpha}.$$

The coordinate expressions of Euler section C and $\sigma^1(C)$ are

$$C = y^\alpha \mathcal{V}_\alpha, \quad \sigma^1(C) = y^\alpha \frac{\partial}{\partial y^\alpha},$$

and the local expression of J is given by

$$J = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha, \quad (1.3.5)$$

where $\{\mathcal{X}^\alpha, \mathcal{V}_\alpha\}$ denotes the corresponding dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$.

The Nijenhuis tensor of the vertical endomorphism

$$\mathbf{N}_J((z, w) = [Jz, Jw]_E - J[Jz, w]_E - J[z, Jw]_E + J^2[z, w]_E$$

vanishes, and it results that J is a integrable structure.

The expression of the complete lift of a section $\rho = \rho^\alpha s_\alpha$ is

$$\rho^c = \rho^\alpha \mathcal{X}_\alpha + (\sigma_\varepsilon^i \frac{\partial \rho^\alpha}{\partial x^i} - L_{\beta\varepsilon}^\alpha \rho^\beta) y^\varepsilon \mathcal{V}_\alpha,$$

and therefore

$$\sigma^1(\rho^c) = \rho^\alpha \sigma_\alpha^i \frac{\partial}{\partial x^i} + \left(\sigma_\varepsilon^i \frac{\partial \rho^\alpha}{\partial x^i} - L_{\beta\varepsilon}^\alpha \rho^\beta \right) y^\varepsilon \frac{\partial}{\partial y^\alpha}.$$

In particular

$$s_\alpha^v = \mathcal{V}_\alpha, \quad s_\alpha^c = \mathcal{X}_\alpha - L_{\alpha\varepsilon}^\beta y^\varepsilon \mathcal{V}_\beta.$$

The local expression of the differential of a function L on $\mathcal{T}E$ is

$$d^E L = \sigma_\alpha^i \frac{\partial L}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha} \mathcal{V}^\alpha,$$

and therefore, we have

$$d^E x^i = \sigma_\alpha^i \mathcal{X}^\alpha, \quad d^E y^\alpha = \mathcal{V}^\alpha.$$

The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d^E \mathcal{X}^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d^E \mathcal{V}^\alpha = 0. \quad (1.3.6)$$

In local coordinates a semispray has the expression

$$\mathcal{S}(x, y) = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha(x, y) \mathcal{V}_\alpha. \quad (1.3.7)$$

The integral curves of $\sigma^1(\mathcal{S})$ satisfy the differential equations

$$\frac{dx^i}{dt} = \sigma_\alpha^i(x) y^\alpha, \quad \frac{dy^\alpha}{dt} = \mathcal{S}^\alpha(x, y). \quad (1.3.8)$$

If we have the relation

$$[C, \mathcal{S}]_{\mathcal{T}E} = \mathcal{S},$$

then \mathcal{S} is called a spray and the functions \mathcal{S}^α are homogeneous functions of degree 2 in y^α .

Under a change of coordinates $x^{i'} = x^{i'}(x^i)$, $y^{\alpha'} = A_{\alpha}^{\alpha'}(x^i)y^\alpha$ on E the transformation rule of the coordinates on $\mathcal{T}E$ is given by

$$\begin{aligned} x^{i'} &= x^{i'}(x^i), \\ y^{\alpha'} &= A_{\alpha}^{\alpha'}(x^i)y^\alpha, \\ u^{\alpha'} &= A_{\alpha}^{\alpha'}(x^i)u^\alpha, \\ \nu^{\alpha'} &= A_{\alpha}^{\alpha'}\nu^\alpha + \sigma_{\alpha}^i u^\alpha \frac{\partial A_{\beta}^{\alpha'}}{\partial x^i} y^\beta. \end{aligned}$$

and the rule of change of base is

$$\begin{aligned} \mathcal{X}_{\alpha} &= A_{\alpha}^{\alpha'} \mathcal{X}_{\alpha'} + \sigma_{\alpha}^i \frac{\partial A_{\beta}^{\alpha'}}{\partial x^i} y^\beta \mathcal{V}_{\alpha'}, \\ \mathcal{V}_{\alpha} &= A_{\alpha}^{\alpha'} \mathcal{V}_{\alpha'}. \end{aligned}$$

The rule of change of dual base has the form

$$\begin{aligned} \mathcal{X}^{\alpha} &= A_{\alpha'}^{\alpha} \mathcal{X}^{\alpha'}, \\ \mathcal{V}^{\alpha} &= A_{\alpha'}^{\alpha} \mathcal{V}^{\alpha'} + \sigma_{\alpha'}^i \frac{\partial A_{\beta}^{\alpha}}{\partial x^i} y^\beta \mathcal{X}^{\alpha'}. \end{aligned}$$

1.3.1 Ehresmann nonlinear connections on Lie algebroid $\mathcal{T}E$

As in the case $E = TM$ [34, 45] we can define the nonlinear connection.

Definition 1.3.4 *An Ehresmann nonlinear connection (or connection) on $\mathcal{T}E$ is an almost product structure \mathcal{N} on $\pi_2 : \mathcal{T}E \rightarrow E$ (i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E \rightarrow \mathcal{T}E$, such that $\mathcal{N}^2 = Id$) smooth on $\mathcal{T}E \setminus \{0\}$ such that*

$$V\mathcal{T}E = \ker(Id + \mathcal{N}).$$

If \mathcal{N} is a connection on $\mathcal{T}E$ then $H\mathcal{T}E = \ker(Id - \mathcal{N})$ is the horizontal subbundle associated to \mathcal{N} and

$$\mathcal{T}E = V\mathcal{T}E \oplus H\mathcal{T}E.$$

Each $\rho \in \Gamma(\mathcal{T}E)$ can be written as $\rho = \rho^h + \rho^v$ where ρ^h, ρ^v are sections in the horizontal and respective vertical subbundles. If $\rho^h = 0$, then ρ is called *vertical* and if $\rho^v = 0$, then ρ is called *horizontal*. A connection \mathcal{N} on $\mathcal{T}E$ induces two projectors $h, v : \mathcal{T}E \rightarrow \mathcal{T}E$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in \Gamma(\mathcal{T}E)$. We have

$$h = \frac{1}{2}(Id + \mathcal{N}), \quad v = \frac{1}{2}(Id - \mathcal{N}), \quad (1.3.9)$$

$$\ker h = \text{Im } v = V\mathcal{T}E, \quad \text{Im } h = \ker v = H\mathcal{T}E.$$

Locally, a connection can be expressed as

$$\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2\mathcal{N}_\alpha^\beta \mathcal{V}_\beta, \quad \mathcal{N}(\mathcal{V}_\beta) = -\mathcal{V}_\beta, \quad (1.3.10)$$

where $\mathcal{N}_\alpha^\beta = \mathcal{N}_\alpha^\beta(x, y)$ are the local coefficients of \mathcal{N} . The sections

$$\delta_\alpha = h(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta \mathcal{V}_\beta, \quad (1.3.11)$$

generate a basis of $H\mathcal{T}E$. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called *adapted*. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where

$$\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - \mathcal{N}_\beta^\alpha \mathcal{X}^\beta.$$

Proposition 1.3.1 *The Lie brackets of the adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ are*

$$[\delta_\alpha, \delta_\beta]_E = L_{\alpha\beta}^\gamma \delta_\gamma + \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \quad [\delta_\alpha, \mathcal{V}_\beta]_E = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} \mathcal{V}_\gamma, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_E = 0, \quad (1.3.12)$$

where

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\beta^i \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial x^i} - \sigma_\alpha^i \frac{\partial \mathcal{N}_\beta^\gamma}{\partial x^i} - \mathcal{N}_\beta^\varepsilon \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\varepsilon} + \mathcal{N}_\alpha^\varepsilon \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\varepsilon} + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma. \quad (1.3.13)$$

Proof. Using (1.2.1) and (1.3.11) we get

$$[\delta_\alpha, \delta_\beta]_{\mathcal{T}E} = \left(\sigma_\beta^i \frac{\partial \mathcal{N}_\alpha^\varepsilon}{\partial x^i} - \mathcal{N}_\beta^\gamma \frac{\partial \mathcal{N}_\alpha^\varepsilon}{\partial y^\gamma} - \sigma_\alpha^i \frac{\partial \mathcal{N}_\beta^\varepsilon}{\partial x^i} + \mathcal{N}_\alpha^\gamma \frac{\partial \mathcal{N}_\beta^\varepsilon}{\partial y^\gamma} \right) \mathcal{V}_\varepsilon + L_{\alpha\beta}^\gamma \mathcal{X}_\gamma.$$

If we insert $\mathcal{X}_\gamma = \delta_\gamma + \mathcal{N}_\gamma^\varepsilon \mathcal{V}_\varepsilon$ then the first relation from (1.3.12) is obtained. By direct computation the second relation is verified. \square

We recall that the Nijenhuis tensor of an endomorphism A is given by

$$\mathbf{N}_A(z, w) = [Az, Aw]_{\mathcal{T}E} - A[Az, w]_{\mathcal{T}E} - A[z, Aw]_{\mathcal{T}E} + A^2[z, w]_{\mathcal{T}E}.$$

Definition 1.3.5 *The curvature of the connection \mathcal{N} on $\mathcal{T}E$ is given by $\Omega = -\mathbf{N}_h$ where h is the horizontal projector and \mathbf{N}_h is the Nijenhuis tensor of h .*

Proposition 1.3.2 *In local coordinates we have*

$$\Omega = -\frac{1}{2} \mathcal{R}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma,$$

where $\mathcal{R}_{\alpha\beta}^\gamma$ are given by (1.3.13) and represent the local coordinate functions of the curvature tensor Ω in the frame $\wedge^2 \mathcal{T}E^* \otimes \mathcal{T}E$ induced by $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}$.

Proof. Since $h^2 = h$ we obtain

$$\Omega(z, w) = -[hz, hw]_{\mathcal{T}E} + h[hz, w]_{\mathcal{T}E} + h[z, hw]_{\mathcal{T}E} - h[z, w]_{\mathcal{T}E},$$

$$\Omega(hz, hw) = -v[hz, hw]_{\mathcal{T}E}, \quad \Omega(hz, vw) = \Omega(vz, vw) = 0,$$

and in local coordinates we get

$$\Omega(\delta_\alpha, \delta_\beta) = -v[\delta_\alpha, \delta_\beta]_{\mathcal{T}E} = -\mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma,$$

which ends the proof. \square

The curvature of the nonlinear connection is an obstruction to the integrability of $H\mathcal{T}E$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E$.

Remark 1.3.1 *$H\mathcal{T}E$ is integrable if and only if the curvature $\Omega = -\mathbf{N}_h$ of the nonlinear connection vanishes.*

Let Ψ a morphism of vector bundles E and \overline{E} . We recall that the connections N on E and \overline{N} on \overline{E} are Ψ -related if

$$\Psi \circ N = \overline{N} \circ \Psi.$$

We consider the connections \mathcal{N} on \mathcal{TE} and \mathbb{N} on TE which are σ^1 -related and a connection N on T^2M which is σ_* -related with \mathbb{N} on TE and $\tilde{\sigma}$ - related with \mathcal{N} on \mathcal{TE} , where $\tilde{\sigma} : \mathcal{TE} \rightarrow T^2M$ is given by $\tilde{\sigma} = \sigma_* \circ \sigma^1$ and $\sigma_* : TE \rightarrow T^2M$ is the tangent application of σ . It follows

$$\mathbb{N} \circ \sigma^1 = \sigma^1 \circ \mathcal{N}, \quad N \circ \sigma_* = \sigma_* \circ \mathbb{N}, \quad N \circ \tilde{\sigma} = \tilde{\sigma} \circ \mathcal{N}, \quad (1.3.14)$$

Let us consider the adapted basis $(\overset{E}{\delta}_i, \frac{\partial}{\partial y^\beta})$ of \mathbb{N} and $(\overset{TM}{\delta}_i, \frac{\partial}{\partial y^j})$ of N given by

$$\overset{E}{\delta}_i = \frac{\partial}{\partial x^i} - \mathbb{N}_i^\beta \frac{\partial}{\partial y^\beta},$$

and

$$\overset{TM}{\delta}_i = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

Therefore, we get

$$\sigma_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} + \frac{\partial \sigma^k}{\partial x^i} \frac{\partial}{\partial y^k}, \quad \sigma_* \left(\frac{\partial}{\partial y^\alpha} \right) = \sigma_\alpha^i \frac{\partial}{\partial y^i}.$$

Theorem 1.3.1 *The following relations hold*

$$\begin{aligned} \sigma^1(\delta_\alpha) &= \sigma_\alpha^i \overset{E}{\delta}_i, \quad \mathcal{N}_\alpha^\beta = \sigma_\alpha^i \mathbb{N}_i^\beta, \\ \sigma_*(\overset{E}{\delta}_i) &= \overset{TM}{\delta}_i, \quad \frac{\partial \sigma^j}{\partial x^i} + N_i^j = \mathbb{N}_i^\beta \sigma_\beta^j, \\ \tilde{\sigma}(\delta_\alpha) &= \sigma_\alpha^i \overset{TM}{\delta}_i, \quad \sigma_\alpha^i \frac{\partial \sigma^j}{\partial x^i} + \sigma_\alpha^i N_i^j = \mathcal{N}_\alpha^\beta \sigma_\beta^j. \end{aligned} \quad (1.3.15)$$

Proof. The first relation from (1.3.14) leads to the relation $\mathbb{N}(\sigma^1(\delta_\alpha)) = \sigma^1(\delta_\alpha)$ from which we get $\sigma^1(\delta_\alpha) = \sigma_\alpha^i \overset{E}{\delta}_i$ and $\mathcal{N}_\alpha^\beta = \sigma_\alpha^i \mathbb{N}_i^\beta$. In the similar way the others relations are obtained. \square

Proposition 1.3.3 *For the curvature tensors of σ^1 -related connections \mathcal{N} and \mathbb{N} we have the relation*

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\alpha^i \sigma_\beta^j \mathbb{R}_{ij}^\gamma, \quad (1.3.16)$$

where

$$\mathbb{R}_{ij}^\gamma = \overset{E}{\delta}_i (\mathbb{N}_j^\gamma) - \overset{E}{\delta}_j (\mathbb{N}_i^\gamma),$$

is the curvature tensor of the nonlinear connection on TE .

Proof. Using the relation $\mathcal{N}_\alpha^\varepsilon = \sigma_\alpha^i \mathbb{N}_i^\varepsilon$ we obtain

$$\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\beta^j \sigma_\alpha^i \left(\overset{E}{\delta}_j (\mathbb{N}_i^\gamma) - \overset{E}{\delta}_i (\mathbb{N}_j^\gamma) \right) + \mathbb{N}_j^\gamma \left(\sigma_\beta^i \frac{\partial \sigma_\alpha^j}{\partial x^i} - \sigma_\alpha^i \frac{\partial \sigma_\beta^j}{\partial x^i} \right) + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma,$$

and from structure equations of the Lie algebroid (1.2.10), the second term is $\mathbb{N}_j^\gamma \sigma_\varepsilon^j L_{\beta\alpha}^\varepsilon = -\mathcal{N}_\varepsilon^\gamma L_{\alpha\beta}^\varepsilon$, which concludes the proof. \square

Remark 1.3.2 A σ^1 -related connection \mathbb{N} on TE determines a connection \mathcal{N} on \mathcal{TE} with the coefficients $\mathcal{N}_\alpha^\beta = \sigma_\alpha^i \mathbb{N}_i^\beta$ and the curvature $\mathcal{R}_{\alpha\beta}^\gamma = \sigma_\alpha^i \sigma_\beta^j \mathbb{R}_{ij}^\gamma$. The converse is not true because σ is only injective.

Let J be the vertical endomorphism.

Remark 1.3.3 Let \mathcal{N} be a bundle morphism of $\pi_2 : \mathcal{TE} \rightarrow E$, smooth on $\mathcal{TE} \setminus \{0\}$. Then \mathcal{N} is a connection on \mathcal{TE} if and only if

$$J\mathcal{N} = J, \quad \mathcal{N}J = -J.$$

The proof proceeds as in the case $E = TM$ and will be omitted.

Definition 1.3.6 The torsion of a nonlinear connection \mathcal{N} is the vector valued two form $t = [J, \mathbf{h}]$ where \mathbf{h} is the horizontal projector and $[\cdot, \cdot]$ is the Frölicher-Nijenhuis bracket.

Definition 1.3.7 t is a semibasic vector-valued form. Its local expression is

$$t = \frac{1}{2} t_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{V}_\gamma, \quad (1.3.17)$$

where

$$t_{\alpha\beta}^\gamma = \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} - \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} - L_{\alpha\beta}^\gamma. \quad (1.3.18)$$

Proof. We have

$$\begin{aligned} [J, \mathbf{h}](z, w) &= [Jz, \mathbf{h}w]_{\mathcal{TE}} + [\mathbf{h}z, Jw]_{\mathcal{TE}} + J[z, w]_{\mathcal{TE}} - J[z, \mathbf{h}w]_{\mathcal{TE}} - \\ &\quad - J[\mathbf{h}z, w]_{\mathcal{TE}} - \mathbf{h}[z, Jw]_{\mathcal{TE}} - \mathbf{h}[Jz, w]_{\mathcal{TE}}, \end{aligned}$$

and in local coordinates we get

$$t(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \left(\frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} - \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} - L_{\alpha\beta}^\gamma \right) \mathcal{V}_\gamma, \quad t(\mathcal{X}_\alpha, \mathcal{V}_\beta) = t(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0.$$

□

Now, let us consider the linear mapping $\mathbb{F} : \mathcal{TE} \rightarrow \mathcal{TE}$, defined by

$$\mathbb{F}(\mathbf{h}z) = -\mathbf{v}z, \quad \mathbb{F}(\mathbf{v}z) = \mathbf{h}z, \quad (1.3.19)$$

for $z \in \Gamma(\mathcal{TE})$ and \mathbf{h}, \mathbf{v} the horizontal and vertical projectors of the nonlinear connection on \mathcal{TE} .

Proposition 1.3.4 The mapping \mathbb{F} has the properties:

- i) \mathbb{F} is globally defined on \mathcal{TE} ,
- ii) Locally, it is given by

$$\mathbb{F} = -\mathcal{V}_\alpha \otimes \mathcal{X}^\alpha + \delta_\alpha \otimes \delta \mathcal{V}^\alpha, \quad (1.3.20)$$

- iii) \mathbb{F} is an almost complex structure $\mathbb{F} \circ \mathbb{F} = -Id$.

Proof. It results by definition that \mathbb{F} is globally defined and

$$(\mathbb{F} \circ \mathbb{F})(hz) = \mathbb{F}(-vz) = -hz, \quad (\mathbb{F} \circ \mathbb{F})(vz) = \mathbb{F}(hz) = -vz.$$

In local coordinates we get $\mathbb{F}(\delta_\alpha) = -\mathcal{V}_\alpha$ and $\mathbb{F}(\mathcal{V}_\alpha) = \delta_\alpha$ which ends the proof. \square

Proposition 1.3.5 *The almost complex structure is integrable if and only if the nonlinear connection is locally flat, that is the curvature and torsion vanish.*

Proof. Let $\mathbf{N}_\mathbb{F}$ be the Nijenhuis tensor of the almost complex structure. We find

$$\begin{aligned} \mathbf{N}_\mathbb{F}(\delta_\alpha, \delta_\beta) &= t_{\alpha\beta}^\gamma \delta_\gamma - \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \\ \mathbf{N}_\mathbb{F}(\delta_\alpha, \mathcal{V}_\beta) &= -\mathcal{R}_{\alpha\beta}^\gamma \delta_\gamma - t_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \\ \mathbf{N}_\mathbb{F}(\mathcal{V}_\alpha, \mathcal{V}_\beta) &= -\mathbf{N}_\mathbb{F}(\delta_\alpha, \delta_\beta). \end{aligned} \tag{1.3.21}$$

From (1.3.21) one reads immediately that $\mathbf{N}_\mathbb{F} = 0$ if and only if $t = 0$ and $\Omega = 0$. \square

A curve $u : [t_0, t_1] \rightarrow E$ is called *admissible* if $\sigma(u(t)) = \dot{c}(t)$ where $c(t) = \pi(u(t))$ is the base curve. A nonlinear connection on $\mathcal{T}E$ induces a covariant derivative of the sections defined locally as follows

$$\mathcal{D}_\rho \eta = \rho^\alpha \left(\sigma_\alpha^i \frac{\partial \eta^\beta}{\partial x^i} + \mathcal{N}_\alpha^\beta \right) s_\beta,$$

where $\rho = \rho^\alpha s_\alpha$ and $\eta = \eta^\alpha s_\alpha$. The derivative is linear in the first argument and it respects multiplication of second argument by real numbers, but not necessarily sum, except the case when the coefficients \mathcal{N}_α^β are the local coefficients of a linear connection. The linearity in the first argument permits us to define the derivative of a section $\eta \in \Gamma(E)$ with respect to $a \in E_u$ by setting

$$\mathcal{D}_a \eta = (\mathcal{D}_\rho \eta)(u),$$

where $\rho \in \Gamma(E)$ is satisfying $\rho(u) = a$. Also, the covariant derivative allows us to take the derivative of sections along curves. If we have a morphism of Lie algebroids $\Phi : F \rightarrow E$ over the map $\varphi : N \rightarrow M$ and a section $\eta : N \rightarrow E$ along φ , i.e $\eta(n) \in E_{\varphi(n)}$, $n \in N$, then η can be written in the form

$$\eta = \sum_{l=1}^p F_l(\xi_l \circ \varphi),$$

for some sections $\{\xi_1, \dots, \xi_p\}$ of E and some functions $\{F_1, \dots, F_p\} \in C^\infty(N)$ and the derivative of η along φ is given by

$$\mathcal{D}_b \eta = \sum_{l=1}^p [(\sigma_F(b) F_l) \xi_l(\varphi(n)) + F_l(n) \mathcal{D}_{\Phi(b)} \xi_l], \quad b \in F_n,$$

where σ_F is the anchor map of the Lie algebroid $F \rightarrow N$ (see [30]).

Let $a : I \rightarrow E$ be an admissible curve and let $b : I \rightarrow E$ be a curve in E , both of them projecting by π onto the same curve γ in M , $\pi(a(t)) = \pi(b(t)) = \gamma(t)$. Take the particular case of Lie algebroid structure $TI \rightarrow I$ and the morphism $\Phi : TI \rightarrow E$, $\Phi(t, \dot{t}) = \dot{t} \gamma(t)$ over $\gamma : I \rightarrow M$. Then one can define the derivative of $b(t)$ along $a(t)$ as $\mathcal{D}_{d/dt} b(t)$. In local coordinates, we obtain

$$\mathcal{D}_{a(t)} b(t) = \left(\frac{db^\beta}{dt} + \mathcal{N}_\alpha^\beta a^\alpha \right) s_\beta(\gamma(t)).$$

Definition 1.3.8 *An admissible curve $c(t)$ is a path (autoparallel) for nonlinear connection \mathcal{N} if and only if*

$$\mathcal{D}_{c(t)} c(t) = 0.$$

In local coordinates we get

$$\frac{dc^\beta}{dt} + \mathcal{N}_\alpha^\beta(x, y) c^\alpha = 0.$$

From the previous considerations we have:

Proposition 1.3.6 *An admissible curve $c(t)$ in E is autoparallel for the nonlinear connection if and only if*

$$\frac{dx^i}{dt} = \sigma_\alpha^i y^\alpha, \quad \frac{dy^\beta}{dt} + \mathcal{N}_\alpha^\beta y^\alpha = 0, \quad (1.3.22)$$

where $x^i = x^i(t) = x^i(c(t))$, $y^\alpha = y^\alpha(t) = y^\alpha(c(t))$, $\sigma_\alpha^i = \sigma_\alpha^i(t) = \sigma_\alpha^i(c(t))$.

Let \mathcal{N} be a nonlinear connection on $\mathcal{T}E$, \mathcal{S}' an arbitrary semispray on $\mathcal{T}E$ and h the horizontal projector of \mathcal{N} . We consider $\mathcal{S} = h\mathcal{S}'$ and for any other semispray \mathcal{S}'' on $\mathcal{T}E$ we have $h(\mathcal{S}' - \mathcal{S}'') = h((\mathcal{S}'^\alpha - \mathcal{S}''^\alpha)\mathcal{V}_\alpha) = 0$ and it results that \mathcal{S} does not depend on the choose of \mathcal{S}' . We have

$$J\mathcal{S} = Jh\mathcal{S}' = J\mathcal{S}' = C,$$

so \mathcal{S} is a semispray, which is called the *associated semispray* to \mathcal{N} .

Proposition 1.3.7 *A nonlinear connection \mathcal{N} and its associated semispray have the same paths.*

Proof. For the arbitrary semispray $\mathcal{S}' = y^\alpha \mathcal{X}_\alpha + \mathcal{S}'^\alpha \mathcal{V}_\alpha$, the associated semispray of \mathcal{N} is

$$\mathcal{S} = h\mathcal{S}' = y^\alpha \mathcal{X}_\alpha - \mathcal{N}_\alpha^\beta y^\alpha \mathcal{V}_\beta,$$

so

$$\mathcal{S}^\beta = -\mathcal{N}_\alpha^\beta y^\alpha.$$

From (1.3.8) and (1.3.22) it results the conclusion. \square

Remark 1.3.4 If \mathcal{S} is a semispray on $\mathcal{T}E$, then we have

$$J[\mathcal{S}, Jz]_{\mathcal{T}E} = -Jz, \quad z \in \Gamma(\mathcal{T}E). \quad (1.3.23)$$

Theorem 1.3.2 Let J be the vertical endomorphism on $\mathcal{T}E$. If ξ is a semispray then

$$\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J, \quad (1.3.24)$$

is a connection on $\mathcal{T}E$.

Proof. Since

$$\mathcal{N}(v) = -\mathcal{L}_{\mathcal{S}}J(v) = -[\mathcal{S}, Jv]_{\mathcal{T}E} + J[\mathcal{S}, v]_{\mathcal{T}E}$$

using (1.3.23) we get

$$J\mathcal{N}(v) = -J[\mathcal{S}, Jv]_{\mathcal{T}E} + J^2[\mathcal{S}, v]_{\mathcal{T}E} = Jv,$$

and

$$\mathcal{N}J(v) = -[\mathcal{S}, J^2v]_{\mathcal{T}E} + J[\mathcal{S}, Jv]_{\mathcal{T}E} = -Jv.$$

By using the Remark 1.3.3 we get the proof of the theorem. \square

Remark 1.3.5 The connection $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$ is induced by the semispray \mathcal{S} . Its local coefficients are given by

$$\mathcal{N}_{\alpha}^{\beta} = \frac{1}{2} \left(-\frac{\partial \mathcal{S}^{\beta}}{\partial y^{\alpha}} + y^{\varepsilon} L_{\alpha\varepsilon}^{\beta} \right). \quad (1.3.25)$$

Proof. By direct computation it results

$$\begin{aligned} \mathcal{N}(\mathcal{X}_{\alpha}) &= -[\mathcal{S}, J(\mathcal{X}_{\alpha})]_{\mathcal{T}E} + J[\mathcal{S}, \mathcal{X}_{\alpha}]_{\mathcal{T}E} \\ &= \mathcal{X}_{\alpha} + \frac{\partial \mathcal{S}^{\beta}}{\partial y^{\alpha}} \mathcal{V}_{\beta} + J(y^{\beta} L_{\beta\alpha}^{\gamma} \mathcal{X}_{\gamma} - \sigma_{\alpha}^i \frac{\partial \mathcal{S}^{\beta}}{\partial x^i} \mathcal{V}_{\beta}) \\ &= \mathcal{X}_{\alpha} + \left(\frac{\partial \mathcal{S}^{\beta}}{\partial y^{\alpha}} + y^{\gamma} L_{\gamma\alpha}^{\beta} \right) \mathcal{V}_{\beta}, \end{aligned}$$

and using (1.3.10) we obtain (1.3.25). \square

Proposition 1.3.8 The torsion of the connection $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$ vanishes.

Proof. We have

$$t = [J, h]_{\mathcal{T}E} = \frac{1}{2} ([J, Id]_{\mathcal{T}E} + [J, -[\xi, J]_{\mathcal{T}E}]_{\mathcal{T}E}) = \frac{1}{2} [J, [J, \xi]_{\mathcal{T}E}]_{\mathcal{T}E}.$$

Using Jacobi identity we obtain that $t = 0$. Also, if we use (1.3.25) into (1.3.18), by direct computation, the same result is obtained. \square

Proposition 1.3.9 *The associated semispray of $\mathcal{N} = -\mathcal{L}_{\mathcal{S}}J$ is given by*

$$\frac{1}{2}(\mathcal{S} - [\mathcal{S}, \mathcal{C}]_{\mathcal{T}E}).$$

Proof. The associated semispray is

$$\mathbf{h}\mathcal{S} = \frac{1}{2}\mathcal{S} + \frac{1}{2}\mathcal{N}(\mathcal{S}) = \frac{1}{2}(\mathcal{S} - [\mathcal{S}, J\mathcal{S}]_{\mathcal{T}E} + J[\mathcal{S}, \mathcal{S}]_{\mathcal{T}E}) = \frac{1}{2}(\mathcal{S} - [\mathcal{S}, \mathcal{C}]_{\mathcal{T}E}).$$

□

1.3.2 Lagrangian formalism on Lie algebroids

We consider the Cartan 1-section

$$\theta_L = J(d^E L),$$

which, in local coordinates is

$$\theta_L = \frac{\partial L}{\partial y^\alpha} \mathcal{X}^\alpha, \quad (1.3.26)$$

The differential of θ_L is the Cartan 2-section

$$\omega_L = d^E \theta_L,$$

and from the local coordinate expression of θ_L we get

$$\omega_L = d^E \left(\frac{\partial L}{\partial y^\alpha} \right) \wedge \mathcal{X}^\alpha + \frac{\partial L}{\partial y^\alpha} \wedge d^E \mathcal{X}^\alpha.$$

But

$$d^E \mathcal{X}^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha \mathcal{X}^\beta \wedge \mathcal{X}^\gamma,$$

and it results [70]

$$\omega_L = \frac{\partial L}{\partial y^\alpha \partial y^\beta} \mathcal{Y}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial L}{\partial y^\gamma} L_{\alpha\beta}^\gamma \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta. \quad (1.3.27)$$

The function L is said to be a regular Lagrangian if ω_L is regular at every point as a bilinear form. Let us consider the energy function given by

$$E_L \stackrel{def}{=} y^\alpha \frac{\partial L}{\partial y^\alpha} - L,$$

and the symplectic equation

$$i_S \omega_L = -d^E E_L, \quad S \in \Gamma(\mathcal{T}E). \quad (1.3.28)$$

In local coordinates, considering the section

$$S = f^\alpha \mathcal{X}_\alpha + S^\alpha \mathcal{Y}_\alpha,$$

we obtain the equations

$$i_S \omega_L = \left(S^\beta g_{\alpha\beta} + f^\beta \left(\sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} + \frac{\partial L}{\partial y^\gamma} L_{\alpha\beta}^\gamma \right) \right) \mathcal{X}^\alpha - f^\beta g_{\alpha\beta} \mathcal{Y}^\alpha,$$

$$-dE_L = \left(\sigma_\alpha^i \frac{\partial L}{\partial x^i} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\beta \right) \mathcal{X}^\alpha - y^\beta g_{\alpha\beta} \mathcal{V}^\alpha.$$

The equality of the \mathcal{V}^α components yields

$$g_{\alpha\beta}(y^\beta - f^\beta) = 0,$$

and the regularity of the Lagrangian implies $y^\beta = f^\beta$, which means that $\mathcal{S} = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha \mathcal{V}_\alpha$ is a semispray. Analogously, the equality of the \mathcal{X}^α components leads to the equation

$$\mathcal{S}^\beta g_{\alpha\beta} + \sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} y^\beta + \frac{\partial L}{\partial y^\gamma} y^\beta L_{\alpha\beta}^\gamma = \sigma_\alpha^i \frac{\partial L}{\partial x^i},$$

and the regularity condition of the Lagrangian determines the components of the semispray

$$\mathcal{S}^\varepsilon = g^{\varepsilon\beta} \left(\sigma_\beta^i \frac{\partial L}{\partial x^i} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L_{\beta\alpha}^\theta y^\alpha \frac{\partial L}{\partial y^\theta} \right), \quad (1.3.29)$$

where $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$. From (1.3.29) and (1.3.25) it results:

Corollary 1.3.1 *For a regular Lagrangian L , there exists a nonlinear connection \mathcal{N} with the coefficients given by*

$$\mathcal{N}_\alpha^\varepsilon = \frac{1}{2} \left(-\frac{\partial \mathcal{S}^\varepsilon}{\partial y^\alpha} + y^\beta L_{\alpha\beta}^\varepsilon \right), \quad (1.3.30)$$

where

$$\mathcal{S}^\varepsilon = g^{\varepsilon\beta} \left(\sigma_\beta^i \frac{\partial L}{\partial x^i} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L_{\beta\alpha}^\theta y^\alpha \frac{\partial L}{\partial y^\theta} \right),$$

and will be called the *canonical nonlinear connection* induced by a regular Lagrangian L .

If (x^i) are coordinates on M , $\{s_\alpha\}$ is a local basis of $\Gamma(E)$, (x^i, y^α) are the corresponding coordinates on E and $\gamma(t) = (x^i(t), y^\alpha(t))$ then, γ is a solution of the Euler-Lagrange equations if and only if [125]

$$\frac{dx^i}{dt} = \sigma_\alpha^i y^\alpha, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^\alpha} \right) = \sigma_\alpha^i \frac{\partial L}{\partial x^i} - L_{\alpha\beta}^\theta y^\beta \frac{\partial L}{\partial y^\theta}. \quad (1.3.31)$$

1.3.3 Homogeneous connections

Definition 1.3.9 *The morphism*

$$\mathcal{T} = \frac{1}{2} \mathcal{L}_C \mathcal{N},$$

is called the tension of the nonlinear connection.

In local coordinates we get

$$\mathcal{T}(\mathcal{X}_\alpha) = \left(\mathcal{N}_\alpha^\beta - \frac{\partial \mathcal{N}_\alpha^\beta}{\partial y^\gamma} y^\gamma \right) \mathcal{V}_\beta, \quad \mathcal{T}(\mathcal{V}_\alpha) = 0,$$

and it results

$$\mathcal{T} = \left(\mathcal{N}_\alpha^\beta - \frac{\partial \mathcal{N}_\alpha^\beta}{\partial y^\gamma} y^\gamma \right) \mathcal{X}^\alpha \otimes \mathcal{V}_\beta. \quad (1.3.32)$$

It is obvious that \mathcal{T} is vanishing, if and only if the nonlinear connection is homogeneous of degree 1 with respect to y^α .

Proposition 1.3.10 *If \mathcal{S} is a spray then $\mathcal{N} = -\mathcal{L}_\mathcal{S} J$ is a homogeneous nonlinear connection.*

Proof. Using (1.3.25) we get

$$\mathcal{T} = \left(-\frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} + y^\beta \frac{\partial^2 \mathcal{S}^\gamma}{\partial y^\alpha \partial y^\beta} \right) \mathcal{X}^\alpha \otimes \mathcal{V}_\gamma.$$

But \mathcal{S} is a spray and it results that \mathcal{S}^γ is homogeneous of degree 2, there is

$$2\mathcal{S}^\gamma = y^\beta \frac{\partial \mathcal{S}^\gamma}{\partial y^\beta},$$

and

$$\frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} = y^\beta \frac{\partial^2 \mathcal{S}^\gamma}{\partial y^\alpha \partial y^\beta},$$

therefore, the tension vanishes. \square

Definition 1.3.10 *The strong torsion T of \mathcal{N} is given by*

$$T = i_\xi t - \mathcal{T},$$

where \mathcal{T} is the tension, t is the torsion of \mathcal{N} , and i_ξ is the contraction with ξ .

Locally, we obtain

$$T(\delta_\alpha) = \left(\frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} y^\beta - \mathcal{N}_\alpha^\gamma + y^\beta L_{\alpha\beta}^\gamma \right) \mathcal{V}_\gamma, \quad T(\mathcal{V}_\alpha) = 0.$$

Proposition 1.3.11 *The strong torsion T of a nonlinear \mathcal{N} connection vanishes if and only if the torsion and the tension of \mathcal{N} vanish.*

Proof. If $T = 0$ then we have

$$N_\alpha^\gamma = \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} y^\beta + y^\beta L_{\alpha\beta}^\gamma$$

and it results

$$\begin{aligned} t(\delta_\alpha, \delta_\beta) &= \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} + \frac{\partial^2 \mathcal{N}_\varepsilon^\gamma}{\partial y^\alpha \partial y^\beta} y^\varepsilon - \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} - \frac{\partial^2 \mathcal{N}_\varepsilon^\gamma}{\partial y^\alpha \partial y^\beta} y^\varepsilon + L_{\alpha\beta}^\gamma - L_{\beta\alpha}^\gamma - L_{\alpha\beta}^\gamma \\ &= \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} - \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\beta} - L_{\alpha\beta}^\gamma = -t(\delta_\alpha, \delta_\beta). \end{aligned}$$

which yields $t(\delta_\alpha, \delta_\beta) = 0$ and $\mathcal{T} = 0$. \square

Definition 1.3.11 *A function $\mathcal{F} : E \rightarrow [0, \infty]$ which satisfies the following properties*

- 1) \mathcal{F} is C^∞ on $E \setminus \{0\}$
- 2) $\mathcal{F}(\lambda u) = \lambda \mathcal{F}(u)$ for $\lambda > 0$ and $u \in E_x$. $x \in M$.
- 3) For each $y \in E_x \setminus \{0\}$ the quadratic form

$$g_{\alpha\beta}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{F}^2}{\partial y^\alpha \partial y^\beta},$$

is positive definite, will be called the Finsler function on a Lie algebroid.

If we insert $\mathcal{L} = \frac{1}{2} \mathcal{F}^2 = \frac{1}{2} g_{\alpha\beta} y^\alpha y^\beta$ into the expression of semispray (1.3.29) we obtain [95]

Corollary 1.3.2 *A homogeneous nonlinear connections has the coefficients given by*

$$\mathcal{N}_\alpha^\varepsilon = \frac{1}{2} \left(-\frac{\partial \mathcal{S}^\varepsilon}{\partial y^\alpha} + y^\beta L_{\alpha\beta}^\varepsilon \right),$$

with

$$\mathcal{S}^\delta = \frac{1}{2} g^{\delta\beta} \left(\sigma_\alpha^i \frac{\partial g_{\beta\gamma}}{\partial x^i} + \sigma_\gamma^i \frac{\partial g_{\alpha\beta}}{\partial x^i} - \sigma_\beta^i \frac{\partial g_{\alpha\gamma}}{\partial x^i} + g_{\varepsilon\alpha} L_{\beta\gamma}^\varepsilon + g_{\varepsilon\gamma} L_{\beta\alpha}^\varepsilon - g_{\varepsilon\beta} L_{\gamma\alpha}^\varepsilon \right) y^\alpha y^\gamma. \quad (1.3.33)$$

and is called the canonical nonlinear connection associated to a Finsler function.

Remark 1.3.6 *In the particular case of the standard Lie algebroid $E = TM$ and $\sigma = Id$ the Cartan nonlinear connection is obtained.*

We consider the canonical nonlinear connection and $\|y\|^2 = g_{\alpha\beta}y^\alpha y^\beta = \mathcal{F}^2$ is the square of the norm of the *Euler* section. The almost complex structure characterized by (1.3.20) does not preserve the property of homogeneity of the sections. Indeed, it applies the 1-homogeneous section δ_α onto the 0-homogeneous section \mathcal{V}_α , $\alpha \in \overline{1, m}$. We define a new almost complex structure $\mathbb{F}_0 : \mathcal{T}E \rightarrow \mathcal{T}E$ given by

$$\mathbb{F}_0(\delta_\alpha) = -\frac{\mathcal{F}}{a}\mathcal{V}_\alpha, \quad \mathbb{F}_0(\mathcal{V}_\alpha) = \frac{a}{\mathcal{F}}\delta_\alpha, \quad a > 0.$$

It is not difficult to prove that $\mathbb{F}_0^2 = -Id$ and \mathbb{F}_0 preserves the property of the homogeneity of the sections.

Theorem 1.3.3 *The almost complex structure \mathbb{F}_0 is integrable if and only if the following relations hold*

$$\begin{aligned} \mathcal{R}_{\alpha\beta}^\gamma &= \frac{1}{a^2} \left(y_\alpha \delta_\beta^\gamma - y_\beta \delta_\alpha^\gamma \right), \\ \delta_\alpha(\mathcal{F}^2) \delta_\beta^\gamma &= \delta_\beta(\mathcal{F}^2) \delta_\alpha^\gamma, \end{aligned} \tag{1.3.34}$$

where $y_\alpha = g_{\alpha\beta}y^\beta$, $\alpha, \beta, \gamma = \overline{1, m}$.

Proof. For the Nijenjuis tensor $\mathbf{N}_{\mathbb{F}_0}$ we have

$$\begin{aligned} \mathbf{N}_{\mathbb{F}_0}(\delta_\alpha, \delta_\beta) &= \left(t_{\alpha\beta}^\gamma + \frac{1}{2\mathcal{F}^2} \left(\delta_\beta(\mathcal{F}^2) \delta_\alpha^\gamma - \delta_\alpha(\mathcal{F}^2) \delta_\beta^\gamma \right) \right) \delta_\gamma + \\ &\quad + \left(\frac{1}{a^2} (y_\alpha \delta_\beta^\gamma - y_\beta \delta_\alpha^\gamma) - \mathcal{R}_{\alpha\beta}^\gamma \right) \mathcal{V}_\gamma, \\ \mathbf{N}_{\mathbb{F}_0}(\delta_\alpha, \mathcal{V}_\beta) &= \left(\frac{1}{\mathcal{F}^2} (y_\alpha \delta_\beta^\gamma - y_\beta \delta_\alpha^\gamma) - \frac{a^2}{\mathcal{F}^2} \mathcal{R}_{\alpha\beta}^\gamma \right) \delta_\gamma - \\ &\quad - \left(t_{\alpha\beta}^\gamma + \frac{1}{2\mathcal{F}^2} \left(\delta_\beta(\mathcal{F}^2) \delta_\alpha^\gamma - \delta_\alpha(\mathcal{F}^2) \delta_\beta^\gamma \right) \right) \mathcal{V}_\gamma, \\ \mathbf{N}_{\mathbb{F}_0}(\delta_\alpha, \delta_\beta) &= -\frac{\mathcal{F}^2}{a^2} \mathbf{N}_{\mathbb{F}_0}(\mathcal{V}_\alpha, \mathcal{V}_\beta). \end{aligned}$$

It follows that $\mathbf{N}_{\mathbb{F}_0} = 0$ if and only if the relations (1.3.34) are satisfied. \square

1.3.4 Linear connections on Lie algebroids

A linear connection on a Lie algebroid $(E, [\cdot, \cdot]_E, \sigma)$ is a map

$$\mathcal{D} : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E),$$

which satisfies the rules

- i) $\mathcal{D}_{\rho+\omega}\eta = \mathcal{D}_\rho\eta + \mathcal{D}_\omega\eta$,
 - ii) $\mathcal{D}_\rho(\eta + \omega) = \mathcal{D}_\rho\eta + \mathcal{D}_\rho\omega$
 - iii) $\mathcal{D}_{f\rho}\eta = f\mathcal{D}_\rho\eta$,
 - iv) $\mathcal{D}_\rho(f\eta) = (\sigma(\rho)f)\eta + f\mathcal{D}_\rho\eta$
- for any function $f \in C^\infty(M)$ and $\rho, \eta, \omega \in \Gamma(E)$.

For $\rho, \eta \in \Gamma(E)$ the section $\mathcal{D}_\rho\eta \in \Gamma(E)$ is called the *covariant derivative* of the section η with respect to the section ρ . Let \mathcal{N} be a nonlinear connection. We have:

Definition 1.3.12 *A linear connection \mathcal{D} on Lie algebroids is called \mathcal{N} -linear connection if*

- i) *\mathcal{D} preserves by parallelism the horizontal distribution HTE .*
- ii) *The tangent structure J is absolute parallel with \mathcal{D} , that is $\mathcal{D}J = 0$.*

Consequently, the following properties hold:

$$\begin{aligned} (\mathcal{D}_\rho\eta^h)^\vee &= 0, & (\mathcal{D}_\rho\eta^\vee)^h &= 0, & \mathcal{D}_\rho h &= 0, & \mathcal{D}_\rho v &= 0, \\ \mathcal{D}_\rho(J\eta^h) &= J(\mathcal{D}_\rho\eta^h), & \mathcal{D}_\rho(J\eta^\vee) &= J(\mathcal{D}_\rho\eta^\vee). \end{aligned}$$

If we denote

$$\mathcal{D}_\rho^h\eta = \mathcal{D}_{\rho^h}\eta, \quad \mathcal{D}_\rho^\vee\eta = \mathcal{D}_{\rho^\vee}\eta,$$

then the following decomposition is obtained

$$\mathcal{D}_\rho = \mathcal{D}_\rho^h + \mathcal{D}_\rho^\vee, \quad \rho \in \Gamma(E).$$

We remark that \mathcal{D}^h and \mathcal{D}^\vee are not covariant derivative, because $\mathcal{D}_\rho^h f = \sigma(\rho^h)f \neq \sigma(\rho)f$, $\mathcal{D}_\rho^\vee f = \sigma(\rho^\vee)f \neq \sigma(\rho)f$, but, it still preserves many properties of \mathcal{D} . Indeed, \mathcal{D}^h and \mathcal{D}^\vee satisfy the Leibniz rule, and \mathcal{D}^h and \mathcal{D}^\vee will be called the *h-covariant derivation* and *v-covariant derivation*, respectively. Using the fact that $\mathcal{D}_{\delta_\alpha} = \mathcal{D}_{\delta_\alpha}^h$, $\mathcal{D}_{\mathcal{V}_\alpha} = \mathcal{D}_{\mathcal{V}_\alpha}^\vee$ we get

Proposition 1.3.12 *In the adapted basis $\{\delta_\alpha, \mathcal{V}_\alpha\}$ a \mathcal{N} -linear connection can be uniquely represented in the form*

$$\begin{aligned} \mathcal{D}_{\delta_\beta}^h \delta_\alpha &= F_{\alpha\beta}^\gamma \delta_\gamma, & \mathcal{D}_{\delta_\beta}^\vee \mathcal{V}_\alpha &= F_{\alpha\beta}^\gamma \mathcal{V}_\gamma, \\ \mathcal{D}_{\mathcal{V}_\beta}^h \delta_\alpha &= C_{\alpha\beta}^\gamma \delta_\gamma, & \mathcal{D}_{\mathcal{V}_\beta}^\vee \mathcal{V}_\alpha &= C_{\alpha\beta}^\gamma \mathcal{V}_\gamma. \end{aligned}$$

The system of functions $(F_{\alpha\beta}^\gamma(x, y), C_{\alpha\beta}^\gamma(x, y))$ represent the local coefficients of h–covariant derivation and of v–covariant derivation, respectively. Under a change of coordinates (1.2.13) the coefficients satisfy the rules

$$F_{\alpha\beta}^\gamma = F_{\alpha'\beta'}^{\gamma'} A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} A_{\gamma'}^{\gamma} + \sigma_{\alpha}^i \frac{\partial A_{\beta}^{\gamma'}}{\partial x^i} A_{\gamma'}^{\gamma},$$

$$C_{\alpha\beta}^\gamma = C_{\alpha'\beta'}^{\gamma'} A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} A_{\gamma'}^{\gamma}.$$

Let us consider a d –tensor T in the local adapted basis, given by

$$T = T_{\gamma\varepsilon}^{\alpha\beta} \delta_{\alpha} \otimes \mathcal{V}_{\beta} \otimes \mathcal{X}^{\gamma} \otimes \delta \mathcal{V}^{\varepsilon}$$

A d –tensor means a tensor on $\mathcal{T}E$, whose components, under a change of coordinates on $\mathcal{T}E$ behave like the components of a tensor field on the base manifold E . Its covariant derivative with respect to $\xi = \xi^h + \xi^v = \xi^{\alpha} \delta_{\alpha} + \zeta^{\beta} \mathcal{V}_{\beta}$ is given by

$$\mathcal{D}_{\xi} T = \left(\xi^{\kappa} T_{\gamma\varepsilon/\kappa}^{\alpha\beta} + \zeta^{\nu} T_{\gamma\varepsilon}^{\alpha\beta} /_{\nu} \right) \delta_{\alpha} \otimes \mathcal{V}_{\beta} \otimes \mathcal{X}^{\gamma} \otimes \delta \mathcal{V}^{\varepsilon}$$

where we have the h–covariant derivative $\mathcal{D}_{\xi}^h T = \xi^{\kappa} T_{\gamma\varepsilon/\kappa}^{\alpha\beta} \delta_{\alpha} \otimes \mathcal{V}_{\beta} \otimes \mathcal{X}^{\gamma} \otimes \delta \mathcal{V}^{\varepsilon}$ with

$$T_{\gamma\varepsilon/\kappa}^{\alpha\beta} = \sigma_{\kappa}^i \frac{\partial T_{\gamma\varepsilon}^{\alpha\beta}}{\partial x^i} - \mathcal{N}_{\kappa}^{\tau} \frac{\partial T_{\gamma\varepsilon}^{\alpha\beta}}{\partial y^{\tau}} + F_{\tau\kappa}^{\alpha} T_{\gamma\varepsilon}^{\tau\beta} + F_{\tau\kappa}^{\beta} T_{\gamma\varepsilon}^{\alpha\tau} - F_{\varepsilon\kappa}^{\tau} T_{\gamma\tau}^{\alpha\beta} - F_{\gamma\kappa}^{\tau} T_{\tau\varepsilon}^{\alpha\beta}$$

and $'''$ is the operator of h– covariant derivative.

The v–covariant derivative of T is $\mathcal{D}_{\xi}^v T = \xi^{\kappa} T_{\gamma\varepsilon}^{\alpha\beta} /_{\kappa} \delta_{\alpha} \otimes \mathcal{V}_{\beta} \otimes \mathcal{X}^{\gamma} \otimes \delta \mathcal{V}^{\varepsilon}$ with

$$T_{\gamma\varepsilon}^{\alpha\beta} /_{\kappa} = \frac{\partial T_{\gamma\varepsilon}^{\alpha\beta}}{\partial y^{\kappa}} + C_{\tau\kappa}^{\alpha} T_{\gamma\varepsilon}^{\tau\beta} + C_{\tau\kappa}^{\beta} T_{\gamma\varepsilon}^{\alpha\tau} - C_{\varepsilon\kappa}^{\tau} T_{\gamma\tau}^{\alpha\beta} - C_{\gamma\kappa}^{\tau} T_{\tau\varepsilon}^{\alpha\beta}$$

where $''''$ is the operator of v– covariant derivative.

1.3.5 Torsion and curvature of a \mathcal{N} -linear connection

The torsion tensor of a \mathcal{N} -linear connection is defined as usual

$$T(\xi, \omega) = \mathcal{D}_\xi \omega - \mathcal{D}_\omega \xi - [\xi, \omega]_{\mathcal{T}E}.$$

As in the case of tangent bundle we have:

Proposition 1.3.13 *The torsion of a \mathcal{N} -linear connection is completely determined by the following five components*

$$\begin{cases} hT(h\xi, h\omega) = \mathcal{D}_\xi^h h\omega - \mathcal{D}_\omega^h h\xi - h[h\xi, h\omega]_{\mathcal{T}E}, \\ vT(h\xi, h\omega) = -v[h\xi, h\omega]_{\mathcal{T}E}, \\ hT(h\xi, v\omega) = -\mathcal{D}_\omega^v h\xi - h[h\xi, v\omega]_{\mathcal{T}E}, \\ vT(h\xi, v\omega) = \mathcal{D}_\xi^h v\omega - v[h\xi, v\omega]_{\mathcal{T}E}, \\ vT(v\xi, v\omega) = \mathcal{D}_\xi^v v\omega - \mathcal{D}_\omega^v v\xi - v[v\xi, v\omega]_{\mathcal{T}E}. \end{cases} \quad (1.3.35)$$

With respect to the adapted basis the components of torsion are given by

$$\begin{cases} hT(\delta_\beta, \delta_\alpha) = T_{\alpha\beta}^\gamma \delta_\gamma = \left(F_{\alpha\beta}^\gamma - F_{\beta\alpha}^\gamma - L_{\alpha\beta}^\gamma \right) \delta_\gamma, \\ vT(\delta_\beta, \delta_\alpha) = \mathcal{R}_{\alpha\beta}^\gamma \mathcal{V}_\gamma = \left(\sigma_\beta^i \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial x^i} - \sigma_\alpha^i \frac{\partial \mathcal{N}_\beta^\gamma}{\partial x^i} - \mathcal{N}_\beta^\varepsilon \frac{\partial \mathcal{N}_\alpha^\gamma}{\partial y^\varepsilon} + \mathcal{N}_\alpha^\varepsilon \frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\varepsilon} + L_{\alpha\beta}^\varepsilon \mathcal{N}_\varepsilon^\gamma \right) \mathcal{V}_\gamma, \\ hT(\mathcal{V}_\beta, \delta_\alpha) = C_{\alpha\beta}^\gamma \delta_\gamma, \\ vT(\mathcal{V}_\beta, \delta_\alpha) = P_{\alpha\beta}^\gamma \mathcal{V}_\gamma = \left(\frac{\partial \mathcal{N}_\beta^\gamma}{\partial y^\alpha} - F_{\alpha\beta}^\gamma \right) \mathcal{V}_\gamma, \\ vT(\mathcal{V}_\beta, \mathcal{V}_\alpha) = S_{\alpha\beta}^\gamma \mathcal{V}_\gamma = \left(C_{\alpha\beta}^\gamma - C_{\beta\alpha}^\gamma \right) \mathcal{V}_\gamma. \end{cases}$$

The curvature of a \mathcal{N} -linear connection is defined by

$$R(\xi, \omega)\varphi = \mathcal{D}_\xi \mathcal{D}_\omega \varphi - \mathcal{D}_\omega \mathcal{D}_\xi \varphi - \mathcal{D}_{[\xi, \omega]_{\mathcal{T}E}} \varphi.$$

Proposition 1.3.14 *The tensor of curvature has three essential components*

$$\begin{cases} R(\delta_\gamma, \delta_\beta)\delta_\alpha = R_{\alpha\beta\gamma}^\varepsilon \delta_\varepsilon, \\ R(\mathcal{V}_\gamma, \delta_\beta)\delta_\alpha = P_{\alpha\beta\gamma}^\varepsilon \delta_\varepsilon, \\ R(\mathcal{V}_\gamma, \mathcal{V}_\beta)\delta_\alpha = S_{\alpha\beta\gamma}^\varepsilon \delta_\varepsilon, \end{cases} \quad (1.3.36)$$

given by

$$\begin{aligned} R_{\gamma\beta\alpha}^\delta &= \sigma_\alpha^i \frac{\partial F_{\gamma\beta}^\delta}{\partial x^i} - \mathcal{N}_\alpha^\varepsilon \frac{\partial F_{\gamma\beta}^\delta}{\partial y^\varepsilon} - \sigma_\beta^i \frac{\partial F_{\gamma\alpha}^\delta}{\partial x^i} + \mathcal{N}_\beta^\varepsilon \frac{\partial F_{\gamma\alpha}^\delta}{\partial y^\varepsilon} + \\ &\quad + F_{\rho\alpha}^\delta F_{\gamma\beta}^\rho - F_{\rho\beta}^\delta F_{\gamma\alpha}^\rho - L_{\beta\alpha}^\rho F_{\gamma\rho}^\delta + C_{\gamma\varepsilon}^\delta \mathcal{R}_{\beta\alpha}^\varepsilon. \end{aligned}$$

$$P_{\alpha\beta\gamma}^\gamma = \frac{\partial F_{\alpha\beta}^\gamma}{\partial y^\delta} - \sigma_\beta^i \frac{\partial C_{\alpha\delta}^\gamma}{\partial x^i} + N_\beta^\varepsilon \frac{\partial C_{\alpha\delta}^\gamma}{\partial y^\varepsilon} + F_{\alpha\beta}^\varepsilon C_{\varepsilon\delta}^\gamma - C_{\alpha\delta}^\varepsilon F_{\varepsilon\beta}^\gamma - \frac{\partial N_\beta^\varepsilon}{\partial y^\delta} C_{\alpha\varepsilon}^\delta.$$

$$S_{\gamma\beta\alpha}^\delta = \frac{\partial C_{\gamma\beta}^\delta}{\partial y^\alpha} - \frac{\partial C_{\gamma\alpha}^\delta}{\partial y^\beta} + C_{\rho\alpha}^\delta C_{\gamma\beta}^\rho - C_{\rho\beta}^\delta C_{\gamma\alpha}^\rho.$$

Proposition 1.3.15 *The Ricci identities have the following form*

$$X_{/\beta/\gamma}^\alpha - X_{/\gamma/\beta}^\alpha = R_{\rho\beta\gamma}^\alpha X^\rho - T_{\beta\gamma}^\rho X_{/\rho}^\alpha - \mathcal{R}_{\beta\alpha}^\rho X^\alpha_{/\rho} -$$

$$-L_{\beta\gamma}^\rho \sigma_\rho^i \frac{\partial X^\alpha}{\partial x^i} + \left(\sigma_\gamma^i \frac{\partial \sigma_\beta^j}{\partial x^i} - \sigma_\beta^i \frac{\partial \sigma_\gamma^j}{\partial x^i} \right) \frac{\partial X^\alpha}{\partial x^j},$$

$$X_{/\beta/\gamma}^\alpha - X_{/\gamma/\beta}^\alpha = P_{\rho\beta\gamma}^\alpha X^\rho - C_{\beta\gamma}^\rho X_{/\rho}^\alpha - P_{\beta\gamma}^\rho X^\alpha_{/\rho},$$

$$X_{/\beta/\gamma}^\alpha - X_{/\gamma/\beta}^\alpha = S_{\rho\beta\gamma}^\alpha X^\rho - S_{\beta\gamma}^\rho X^\alpha_{/\rho}.$$

Definition 1.3.13 *A \mathcal{N} -linear connection is called of Cartan type if*

$$\mathcal{D}_\xi^h C = 0, \quad \mathcal{D}_\xi^v C = v\xi, \quad (1.3.37)$$

where $C = y^\alpha \mathcal{V}_\alpha$ is the Euler section.

By direct computation, it results that a \mathcal{N} -linear connection \mathcal{D} is of Cartan type if and only if

$$\mathcal{N}_\beta^\alpha = F_{\varepsilon\beta}^\alpha y^\varepsilon, \quad y^\varepsilon C_{\varepsilon\beta}^\alpha = 0. \quad (1.3.38)$$

Introducing these relations into the coefficients expression of the curvature, we obtain the following result:

Proposition 1.3.16 *A \mathcal{N} -linear connection of Cartan type has the properties*

$$\mathcal{R}_{\beta\gamma}^\alpha = R_{\varepsilon\beta\gamma}^\alpha y^\varepsilon - L_{\beta\gamma}^\alpha, \quad P_{\beta\gamma}^\alpha = P_{\varepsilon\beta\gamma}^\alpha y^\varepsilon, \quad S_{\beta\gamma}^\alpha = S_{\varepsilon\beta\gamma}^\alpha y^\varepsilon. \quad (1.3.39)$$

1.4 Dynamical covariant derivative and metric non-linear connection on Lie algebroid $\mathcal{T}E$

In this section we will introduce the notion of dynamical covariant derivative on Lie algebroids as a tensor derivation and study the compatibility between nonlinear connection and a pseudo-Riemannian metric.

Definition 1.4.1 A map $\nabla : \mathfrak{T}(\mathcal{T}E \setminus \{0\}) \rightarrow \mathfrak{T}(\mathcal{T}E \setminus \{0\})$ is said to be a tensor derivation on $\mathcal{T}E \setminus \{0\}$ if the following conditions are satisfied:

- i) ∇ is \mathbb{R} -linear
- ii) ∇ is type preserving, i.e. $\nabla(\mathfrak{T}_s^r(\mathcal{T}E \setminus \{0\})) \subset \mathfrak{T}_s^r(\mathcal{T}E \setminus \{0\})$, for each $(r, s) \in \mathbb{N} \times \mathbb{N}$.
- iii) ∇ obeys the Leibnitz rule $\nabla(P \otimes S) = \nabla P \otimes S + P \otimes \nabla S$, for any tensors P, S on $\mathcal{T}E \setminus \{0\}$.
- iv) ∇ commutes with any contractions, where $\mathfrak{T}_\bullet(\mathcal{T}E \setminus \{0\})$ is the space of tensors on $\mathcal{T}E \setminus \{0\}$.

For a semispray \mathcal{S} we consider the \mathbb{R} -linear map

$$\nabla_0 : \Gamma(\mathcal{T}E \setminus \{0\}) \rightarrow \Gamma(\mathcal{T}E \setminus \{0\}),$$

given by

$$\nabla_0 \rho = h[\mathcal{S}, h\rho]_{\mathcal{T}E} + v[\mathcal{S}, v\rho]_{\mathcal{T}E}, \quad \forall \rho \in \Gamma(\mathcal{T}E \setminus \{0\}). \quad (1.4.1)$$

It results that

$$\nabla_0(f\rho) = \mathcal{S}(f)\rho + f\nabla_0\rho, \quad \forall f \in C^\infty(E), \rho \in \Gamma(\mathcal{T}E \setminus \{0\}). \quad (1.4.2)$$

Any tensor derivation on $\mathcal{T}E \setminus \{0\}$ is completely determined by its actions on smooth functions and sections on $\mathcal{T}E \setminus \{0\}$ (see [117] generalized Willmore's theorem, p. 1217). Therefore there exists a unique tensor derivation ∇ on $\mathcal{T}E \setminus \{0\}$ such that

$$\nabla|_{C^\infty(E)} = \mathcal{S}, \quad \nabla|_{\Gamma(\mathcal{T}E \setminus \{0\})} = \nabla_0.$$

We will call the tensor derivation ∇ , the *dynamical covariant derivative* induced by the semispray \mathcal{S} and a nonlinear connection N .

Proposition 1.4.1 The following formulas hold

$$[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{T}E} = -\delta_\beta - \left(\mathcal{N}_\beta^\alpha + \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} \right) \mathcal{V}_\alpha, \quad (1.4.3)$$

$$[\mathcal{S}, \delta_\beta]_{\mathcal{T}E} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha + \mathcal{R}_\beta^\gamma \mathcal{V}_\gamma, \quad (1.4.4)$$

where

$$\mathcal{R}_\beta^\gamma = -\sigma_\beta^i \frac{\partial \mathcal{S}^\gamma}{\partial x^i} - \mathcal{S}(\mathcal{N}_\beta^\gamma) + \mathcal{N}_\beta^\alpha \mathcal{N}_\alpha^\gamma + \mathcal{N}_\beta^\alpha \frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} + \mathcal{N}_\varepsilon^\gamma L_{\alpha\beta}^\varepsilon y^\alpha. \quad (1.4.5)$$

The action of the dynamical covariant derivative on the Berwald basis is given by

$$\begin{aligned}\nabla \mathcal{V}_\beta &= \mathbf{v}[\mathcal{S}, \mathcal{V}_\beta]_{TE} = - \left(\mathcal{N}_\beta^\alpha + \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} \right) \mathcal{V}_\alpha \\ \nabla \delta_\beta &= \mathbf{h}[\mathcal{S}, \delta_\beta]_{TE} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha.\end{aligned}$$

It is not difficult to extend the action of ∇ to the algebra of tensors by requiring for ∇ to preserve the tensor product. For a pseudo-Riemannian metric g on VTE (i.e. a $(2,0)$ -type symmetric tensor $g = g_{\alpha\beta}(x, y) \mathcal{V}^\alpha \otimes \mathcal{V}^\beta$ of rank m on VTE) we have

$$(\nabla g)(\rho_1, \rho_2) = \mathcal{S}(g(\rho_1, \rho_2)) - g(\nabla \rho_1, \rho_2) - g(\rho_1, \nabla \rho_2), \quad (1.4.6)$$

and in local coordinates we get

$$g_{\alpha\beta/} := (\nabla g)(\mathcal{V}_\alpha, \mathcal{V}_\beta) = \mathcal{S}(g_{\alpha\beta}) + g_{\gamma\beta} \left(\mathcal{N}_\alpha^\gamma + \frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} \right) + g_{\gamma\alpha} \left(\mathcal{N}_\beta^\gamma + \frac{\partial \mathcal{S}^\gamma}{\partial y^\beta} \right). \quad (1.4.7)$$

Definition 1.4.2 *The nonlinear connection N is called metric or compatible with the metric tensor g if $\nabla g = 0$, that is*

$$\mathcal{S}(g(\rho_1, \rho_2)) = g(\nabla \rho_1, \rho_2) + g(\rho_1, \nabla \rho_2). \quad (1.4.8)$$

If \mathcal{S} be a semispray, \mathcal{N} a nonlinear connection and ∇ the dynamical covariant derivative induced by $(\mathcal{S}, \mathcal{N})$, then we set:

Proposition 1.4.2 *The nonlinear connection $\tilde{\mathcal{N}}$ with the coefficients given by*

$$\tilde{\mathcal{N}}_\beta^\alpha = \mathcal{N}_\beta^\alpha - \frac{1}{2} g^{\alpha\gamma} g_{\gamma\beta/}, \quad (1.4.9)$$

is a metric nonlinear connection.

Proof. Since \mathcal{N}_β^α are the coefficients of a nonlinear connection and $g^{\alpha\gamma} g_{\gamma\beta/}$, are the components of a tensor of type $(1,1)$ it results that $\tilde{\mathcal{N}}_\beta^\alpha$ are also the coefficients of a nonlinear connection. We consider the dynamical covariant derivative induced by $(\mathcal{S}, \tilde{\mathcal{N}})$ and we have

$$\begin{aligned}(\nabla g)(\mathcal{V}_\alpha, \mathcal{V}_\beta) &= \mathcal{S}(g_{\alpha\beta}) + g_{\gamma\beta} \left(\tilde{\mathcal{N}}_\alpha^\gamma + \frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} \right) + g_{\gamma\alpha} \left(\tilde{\mathcal{N}}_\beta^\gamma + \frac{\partial \mathcal{S}^\gamma}{\partial y^\beta} \right) \\ &= \mathcal{S}(g_{\alpha\beta}) + g_{\gamma\beta} \left(\mathcal{N}_\alpha^\gamma + \frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} \right) + g_{\gamma\alpha} \left(\mathcal{N}_\beta^\gamma + \frac{\partial \mathcal{S}^\gamma}{\partial y^\beta} \right) - \\ &\quad - g_{\gamma\beta} \frac{1}{2} g^{\gamma\varepsilon} g_{\varepsilon\alpha/} - g_{\gamma\alpha} \frac{1}{2} g^{\gamma\varepsilon} g_{\varepsilon\beta/} = \\ &= g_{\alpha\beta/} - \frac{1}{2} g_{\alpha\beta/} - \frac{1}{2} g_{\alpha\beta/} = 0,\end{aligned}$$

that is the connection $\tilde{\mathcal{N}}$ is metric. □

For the particular case of the tangent bundle see [19, 21]

1.4.1 The case of SODE connection

A semispray (*SODE*) given by $\mathcal{S} = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha \mathcal{V}_\alpha$ determines an associated nonlinear connection

$$\mathcal{N} = -\mathcal{L}_\mathcal{S} J,$$

with local coefficients (1.3.25)

$$\mathcal{N}_\alpha^\beta = \frac{1}{2} \left(-\frac{\partial \mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L_{\alpha\varepsilon}^\beta \right).$$

Proposition 1.4.3 *The following equations hold*

$$[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{T}E} = -\delta_\beta + (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \mathcal{V}_\alpha, \quad (1.4.10)$$

$$[\mathcal{S}, \delta_\beta]_{\mathcal{T}E} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha + \mathcal{R}_\beta^\alpha \mathcal{V}_\alpha, \quad (1.4.11)$$

where

$$\mathcal{R}_\beta^\alpha = -\sigma_\beta^i \frac{\partial \mathcal{S}^\alpha}{\partial x^i} - \mathcal{S}(\mathcal{N}_\beta^\alpha) - \mathcal{N}_\gamma^\alpha \mathcal{N}_\beta^\gamma + (L_{\varepsilon\beta}^\gamma \mathcal{N}_\gamma^\alpha + L_{\gamma\varepsilon}^\alpha \mathcal{N}_\beta^\gamma) y^\varepsilon. \quad (1.4.12)$$

The dynamical covariant derivative induced by \mathcal{S} and associated nonlinear connection is characterized by

$$\nabla \mathcal{V}_\beta = v[\mathcal{S}, \mathcal{V}_\beta]_{\mathcal{T}E} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \mathcal{V}_\alpha = -\frac{1}{2} \left(\frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} + L_{\beta\varepsilon}^\alpha y^\varepsilon \right) \mathcal{V}_\alpha, \quad (1.4.13)$$

$$\nabla \delta_\beta = h[\mathcal{S}, \delta_\beta]_{\mathcal{T}E} = (\mathcal{N}_\beta^\alpha - L_{\beta\varepsilon}^\alpha y^\varepsilon) \delta_\alpha.$$

$$g_{\alpha\beta/} := (\nabla g)(\mathcal{V}_\alpha, \mathcal{V}_\beta) = \mathcal{S}(g_{\alpha\beta}) - g_{\gamma\beta} \mathcal{N}_\alpha^\gamma - g_{\gamma\alpha} \mathcal{N}_\beta^\gamma + (g_{\gamma\beta} L_{\alpha\varepsilon}^\gamma + g_{\gamma\alpha} L_{\beta\varepsilon}^\gamma) y^\varepsilon, \quad (1.4.14)$$

which is equivalent to

$$(\nabla g)(\mathcal{V}_\alpha, \mathcal{V}_\beta) = \mathcal{S}(g_{\alpha\beta}) + \frac{1}{2} \frac{\partial \mathcal{S}^\gamma}{\partial y^\alpha} g_{\gamma\beta} + \frac{1}{2} \frac{\partial \mathcal{S}^\gamma}{\partial y^\beta} g_{\gamma\alpha} + \frac{1}{2} (g_{\gamma\beta} L_{\alpha\varepsilon}^\gamma + g_{\gamma\alpha} L_{\beta\varepsilon}^\gamma) y^\varepsilon. \quad (1.4.15)$$

Definition 1.4.3 *The Jacobi endomorphism is given by*

$$\Phi = v[\mathcal{S}, h\rho]_{\mathcal{T}E}.$$

Locally, from (1.4.11) we obtain that $\Phi = \mathcal{R}_\beta^\alpha \mathcal{V}_\alpha \otimes \mathcal{X}^\beta$, where \mathcal{R}_β^α is given by (1.4.12) and represent the local coefficients of the Jacobi endomorphism.

Proposition 1.4.4 *The following result holds*

$$\Phi = i_\mathcal{S} \Omega + v[v\mathcal{S}, h\rho]_{\mathcal{T}E}.$$

Proof. Indeed, $\Phi(\rho) = v[\mathcal{S}, h\rho]_{\mathcal{T}E} = v[h\mathcal{S}, h\rho]_{\mathcal{T}E} + v[v\mathcal{S}, h\rho]_{\mathcal{T}E}$ and $\Omega(\mathcal{S}, \rho) = v[h\mathcal{S}, h\rho]_{\mathcal{T}E}$, which yields $\Phi(\rho) = \Omega(\mathcal{S}, \rho) + v[v\mathcal{S}, h\rho]_{\mathcal{T}E}$. \square

If \mathcal{S} is a spray, then the coefficients \mathcal{S}^α are 2-homogeneous with respect to the variables y^β and it results

$$2\mathcal{S}^\alpha = \frac{\partial \mathcal{S}^\alpha}{\partial y^\beta} y^\beta = -2\mathcal{N}_\beta^\alpha y^\beta + L_{\beta\gamma}^\alpha y^\beta y^\gamma = -2\mathcal{N}_\beta^\alpha y^\beta.$$

$$\mathcal{S} = h\mathcal{S} = y^\alpha \delta_\alpha, \quad v\mathcal{S} = 0, \quad \mathcal{N}_\beta^\alpha = \frac{\partial \mathcal{N}_\varepsilon^\alpha}{\partial y^\beta} y^\varepsilon + L_{\beta\varepsilon}^\alpha y^\varepsilon,$$

which yields

$$\Phi = i_{\mathcal{S}}\Omega, \tag{1.4.16}$$

and locally we get

$$\mathcal{R}_\beta^\alpha = \mathcal{R}_{\varepsilon\beta}^\alpha y^\varepsilon, \tag{1.4.17}$$

which represents the local relation between the Jacobi endomorphism and the curvature of the nonlinear connection.

1.4.2 Lagrangian case

Let us consider a regular Lagrangian L on E , that is the matrix

$$g_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta},$$

has constant rank m . The symplectic structure induced by the regular Lagrangian is [70]

$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial L}{\partial y^\gamma} L_{\alpha\beta}^\gamma \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta.$$

Let us consider the energy function given by

$$E_L = y^\alpha \frac{\partial L}{\partial y^\alpha} - L,$$

then the symplectic equation

$$i_S \omega_L = -d^E E_L, \quad S \in \Gamma(\mathcal{T}E),$$

and the regularity condition of the Lagrangian determine the components of the semispray (1.3.29)

$$S^\varepsilon = g^{\varepsilon\beta} \left(\sigma_\beta^i \frac{\partial L}{\partial x^i} - \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\alpha - L_{\beta\alpha}^\theta y^\alpha \frac{\partial L}{\partial y^\theta} \right),$$

where $g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma$.

The connection \mathcal{N} determined by this semispray is the *canonical nonlinear connection* induced by a regular Lagrangian L . Its coefficients are given by

$$\mathcal{N}_\beta^\alpha = \frac{1}{2} g^{\alpha\varepsilon} \left[\mathcal{S}(g_{\varepsilon\beta}) + \sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\varepsilon} - \sigma_\varepsilon^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} - L_{\beta\varepsilon}^\gamma \frac{\partial L}{\partial y^\gamma} + \left(g_{\gamma\varepsilon} L_{\beta\theta}^\gamma + g_{\gamma\beta} L_{\varepsilon\theta}^\gamma \right) y^\theta \right] \quad (1.4.18)$$

Theorem 1.4.1 *The canonical nonlinear connection \mathcal{N} induced by a regular Lagrangian L is a metric nonlinear connection.*

Proof. Introducing the expression of the semispray (1.3.29) into the equation (1.4.15) we obtain

$$\begin{aligned}
(\nabla g)(\mathcal{V}_\alpha, \mathcal{V}_\beta) &= y^\varepsilon \sigma_\varepsilon^i \frac{\partial g_{\alpha\beta}}{\partial x^i} + g^{\varepsilon\gamma} \left(\sigma_\gamma^i \frac{\partial L}{\partial x^i} - \sigma_\theta^i \frac{\partial^2 L}{\partial x^i \partial y^\gamma} y^\theta - L_{\gamma\tau}^\theta y^\tau \frac{\partial L}{\partial y^\theta} \right) \frac{\partial g_{\alpha\beta}}{\partial y^\varepsilon} \\
&\quad + \frac{1}{2} \left(g_{\gamma\beta} \frac{\partial g^{\gamma\varepsilon}}{\partial y^\alpha} + g_{\gamma\alpha} \frac{\partial g^{\gamma\varepsilon}}{\partial y^\beta} \right) \left(\sigma_\varepsilon^i \frac{\partial L}{\partial x^i} - \sigma_\theta^i \frac{\partial^2 L}{\partial x^i \partial y^\varepsilon} y^\theta - L_{\varepsilon\tau}^\theta \frac{\partial L}{\partial y^\theta} y^\tau \right) \\
&\quad + \frac{1}{2} \left(\sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} + \sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} \right) - \sigma_\varepsilon^i \frac{\partial g_{\alpha\beta}}{\partial x^i} y^\varepsilon \\
&\quad - \frac{1}{2} \left(\sigma_\alpha^i \frac{\partial^2 L}{\partial x^i \partial y^\beta} + \sigma_\beta^i \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \right) - \frac{1}{2} \frac{\partial L}{\partial y^\varepsilon} (L_{\beta\alpha}^\varepsilon + L_{\alpha\beta}^\varepsilon) \\
&\quad - \frac{1}{2} \left(g_{\gamma\beta} L_{\alpha\varepsilon}^\gamma + g_{\gamma\alpha} L_{\beta\varepsilon}^\gamma \right) y^\varepsilon + \frac{1}{2} \left(g_{\gamma\beta} L_{\alpha\varepsilon}^\gamma + g_{\gamma\alpha} L_{\beta\varepsilon}^\gamma \right) y^\varepsilon.
\end{aligned}$$

By direct computation, using the equalities

$$g_{\gamma\beta} \frac{\partial g^{\gamma\varepsilon}}{\partial y^\alpha} = -g^{\gamma\varepsilon} \frac{\partial g_{\gamma\beta}}{\partial y^\alpha} = -g^{\gamma\varepsilon} \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}, \quad L_{\alpha\beta}^\theta = -L_{\beta\alpha}^\theta,$$

it results $(\nabla g)(\mathcal{V}_\alpha, \mathcal{V}_\beta) = 0$, which ends the proof. \square

Theorem 1.4.2 *The canonical nonlinear connection induced by a regular Lagrangian is a unique connection which is metric and compatible with the symplectic structure ω_L , that is*

$$\nabla g = 0, \tag{1.4.19}$$

$$\omega_L(\mathbf{h}\rho, \mathbf{h}\nu) = 0, \quad \forall \rho, \nu \in \Gamma(\mathcal{T}E \setminus \{0\}) \tag{1.4.20}$$

Proof. Using the equation $\mathcal{V}^\alpha = \delta\mathcal{V}^\alpha - \mathcal{N}_\beta^\alpha \mathcal{X}^\beta$ it results

$$\begin{aligned}
\omega_L &= g_{\alpha\beta} (\delta\mathcal{V}^\beta - \mathcal{N}_\gamma^\beta \mathcal{X}^\gamma) \wedge \mathcal{X}^\alpha + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon \right) \mathcal{X}^\beta \wedge \mathcal{X}^\alpha \\
&= g_{\alpha\beta} \delta\mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2} \left(g_{\alpha\gamma} \mathcal{N}_\beta^\gamma - g_{\beta\gamma} \mathcal{N}_\alpha^\gamma + \frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon \right) \mathcal{X}^\beta \wedge \mathcal{X}^\alpha \\
&= g_{\alpha\beta} \delta\mathcal{V}^\beta \wedge \mathcal{X}^\alpha + \frac{1}{2} \left(\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} + \frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon \right) \mathcal{X}^\beta \wedge \mathcal{X}^\alpha,
\end{aligned}$$

where $\mathcal{N}_{\alpha\beta} := g_{\alpha\gamma} \mathcal{N}_\beta^\gamma$. We have that $\omega_L(\mathbf{h}\rho, \mathbf{h}\nu) = 0$ if and only if the second part of the above relation vanishes, that is

$$\mathcal{N}_{[\alpha\beta]} = \frac{1}{2} (\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha}) = \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i + \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon \right).$$

It result that the skew symmetric part of $\mathcal{N}_{\alpha\beta}$ is uniquely determined by the condition (1.4.20). The symmetric part of $\mathcal{N}_{\alpha\beta}$ is completely determined by the metric condition (1.4.19). Indeed

$$\begin{aligned}\mathcal{S}(g_{\alpha\beta}) &= g_{\gamma\beta}\mathcal{N}_{\alpha}^{\gamma} + g_{\gamma\alpha}\mathcal{N}_{\beta}^{\gamma} - \left(g_{\gamma\beta}L_{\alpha\varepsilon}^{\gamma} + g_{\gamma\alpha}L_{\beta\varepsilon}^{\gamma}\right)y^{\varepsilon} \\ &= \mathcal{N}_{\beta\alpha} + \mathcal{N}_{\alpha\beta} - \left(g_{\gamma\beta}L_{\alpha\varepsilon}^{\gamma} + g_{\gamma\alpha}L_{\beta\varepsilon}^{\gamma}\right)y^{\varepsilon} \\ &= 2\mathcal{N}_{(\alpha\beta)} - \left(g_{\gamma\beta}L_{\alpha\varepsilon}^{\gamma} + g_{\gamma\alpha}L_{\beta\varepsilon}^{\gamma}\right)y^{\varepsilon}.\end{aligned}$$

that is

$$2\mathcal{N}_{(\alpha\beta)} = \mathcal{S}(g_{\alpha\beta}) + \left(g_{\gamma\beta}L_{\alpha\varepsilon}^{\gamma} + g_{\gamma\alpha}L_{\beta\varepsilon}^{\gamma}\right)y^{\varepsilon}.$$

The equations (1.4.19) and (1.4.20) uniquely determine the coefficients of the nonlinear connection

$$\begin{aligned}\mathcal{N}_{\beta}^{\gamma} &= g^{\gamma\alpha}\mathcal{N}_{\alpha\beta} = g^{\gamma\alpha}(\mathcal{N}_{(\alpha\beta)} + \mathcal{N}_{[\alpha\beta]}) \\ &= \frac{1}{2}g^{\gamma\alpha}\left[\mathcal{S}(g_{\alpha\beta}) + \frac{\partial^2 L}{\partial x^i \partial y^{\alpha}}\sigma_{\beta}^i - \frac{\partial^2 L}{\partial x^i \partial y^{\beta}}\sigma_{\alpha}^i - \frac{\partial L}{\partial y^{\varepsilon}}L_{\beta\alpha}^{\varepsilon}\right] \\ &\quad + \frac{1}{2}g^{\gamma\alpha}\left(g_{\gamma\beta}L_{\alpha\varepsilon}^{\gamma} + g_{\gamma\alpha}L_{\beta\varepsilon}^{\gamma}\right)y^{\varepsilon}\end{aligned}$$

Conversely, introducing (1.3.29) into (1.3.25) we have (1.4.18) which ends the proof.

Remark 1.4.1 *The invariant form of Helmholtz conditions on Lie algebroids is given by:*

$$\begin{aligned}\mathcal{D}_{\rho}^{\nu}g(\nu, \theta) &= \mathcal{D}_{\theta}^{\nu}g(\nu, \rho), \\ \nabla g &= 0, \\ g(\Phi\rho, \nu) &= g(\Phi\nu, \rho),\end{aligned}\tag{1.4.21}$$

for $\nu, \rho, \theta \in \Gamma(E)$, where \mathcal{D}^{ν} is ν -covariant derivative.

In local coordinates we obtain the following equations

$$\begin{aligned}\frac{\partial g_{\alpha\beta}}{\partial y^{\varepsilon}} &= \frac{\partial g_{\alpha\varepsilon}}{\partial y^{\beta}}, \\ \mathcal{S}(g_{\alpha\beta}) - g_{\gamma\beta}N_{\alpha}^{\gamma} - g_{\gamma\alpha}N_{\beta}^{\gamma} &= y^{\varepsilon}\left(g_{\gamma\beta}L_{\varepsilon\alpha}^{\gamma} + g_{\gamma\alpha}L_{\varepsilon\beta}^{\gamma}\right), \\ g_{\alpha\gamma}\left(\sigma_{\beta}^i \frac{\partial \mathcal{S}^{\gamma}}{\partial x^i} + \mathcal{S}N_{\beta}^{\gamma} + N_{\beta}^{\varepsilon}N_{\varepsilon}^{\gamma} - (L_{\varepsilon\beta}^{\delta}N_{\delta}^{\gamma} + L_{\delta\varepsilon}^{\gamma}N_{\beta}^{\delta})y^{\varepsilon}\right) &= \\ g_{\beta\gamma}\left(\sigma_{\alpha}^i \frac{\partial \mathcal{S}^{\gamma}}{\partial x^i} + \mathcal{S}N_{\alpha}^{\gamma} + N_{\alpha}^{\varepsilon}N_{\varepsilon}^{\gamma} - (L_{\varepsilon\alpha}^{\delta}N_{\delta}^{\gamma} + L_{\delta\varepsilon}^{\gamma}N_{\alpha}^{\delta})y^{\varepsilon}\right).\end{aligned}\tag{1.4.22}$$

In the case of standard Lie algebroid $(TM, [\cdot, \cdot], id)$ we obtain the classical Helmholtz conditions [113].

1.5 The prolongation of a Lie algebroid over the vector bundle projection of the dual bundle

Let $\tau : E^* \rightarrow M$ be the dual bundle of $\pi : E \rightarrow M$ and $(E, [\cdot, \cdot]_E, \sigma)$ a Lie algebroid structure over M . One can construct a Lie algebroid structure over E^* , by taking the prolongation of $(E, [\cdot, \cdot]_E, \sigma)$ over $\tau : E^* \rightarrow M$ (see [46], [64], [50]). This structure is given by:

- The associated vector bundle is $(\mathcal{T}E^*, \tau_1, E^*)$ where

$$\mathcal{T}E^* = \cup \mathcal{T}_{u^*} E^*, \quad u^* \in E^*$$

with

$$\mathcal{T}_{u^*} E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*} E^* | \sigma(u_x) = T_{u^*} \tau(v_{u^*}), \tau(u^*) = x \in M\},$$

and the projection $\tau_1 : \mathcal{T}E^* \rightarrow E^*$, $\tau_1(u_x, v_{u^*}) = u^*$.

- The Lie algebra structure $[\cdot, \cdot]_{\mathcal{T}E^*}$ on $\Gamma(\mathcal{T}E^*)$ is defined in the following way: if $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$ are such that $\rho_i(u^*) = (X_i(\tau(u^*)), U_i(u^*))$ where $X_i \in \Gamma(E)$, $U_i \in \chi(E^*)$ and $\sigma(X_i(\tau(u^*))) = T_{u^*} \tau(U_i(u^*))$, $i = 1, 2$, then

$$[\rho_1, \rho_2]_{\mathcal{T}E^*}(u^*) = ([X_1, X_2]_{\mathcal{T}E^*}(\tau(u^*)), [U_1, U_2]_{\mathcal{T}E^*}(u^*)).$$

- The anchor is the projection $\sigma^1 : \mathcal{T}E^* \rightarrow TE^*$, $\sigma^1(u, v) = v$.

Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \rightarrow E$, $\mathcal{T}\tau(u, v) = u$ then $(V\mathcal{T}E^*, \tau_1|_{V\mathcal{T}E^*}, E^*)$ with $V\mathcal{T}E^* = \text{Ker } \mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$, called the *vertical subbundle*. If (q^i, μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \rightarrow M$ then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$ where

$$\mathcal{Q}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma_\alpha^i \frac{\partial}{\partial q^i} |_{u^*} \right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha} |_{u^*} \right). \quad (1.5.1)$$

The structure functions on $\mathcal{T}E^*$ are given by the following formulas

$$\sigma^1(\mathcal{Q}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial q^i}, \quad \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha}, \quad (1.5.2)$$

$$[\mathcal{Q}_\alpha, \mathcal{Q}_\beta]_{\mathcal{T}E^*} = L_{\alpha\beta}^\gamma \mathcal{Q}_\gamma, \quad [\mathcal{Q}_\alpha, \mathcal{P}^\alpha]_{\mathcal{T}E^*} = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = 0, \quad (1.5.3)$$

and therefore

$$\begin{aligned} d^E q^i &= \sigma_\alpha^i \mathcal{Q}^\alpha, \quad d^E \mu_\alpha = \mathcal{P}_\alpha, \\ d^E \mathcal{Q}^\gamma &= -\frac{1}{2} L_{\alpha\beta}^\gamma \mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta, \quad d^E \mathcal{P}_\alpha = 0, \end{aligned} \quad (1.5.4)$$

where $\{\mathcal{Q}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$. Also, if $\rho = \rho^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ is a section of $\mathcal{T}E^*$, then

$$\sigma^1(\rho) = \sigma_\alpha^i \rho^\alpha \frac{\partial}{\partial q^i} + \rho_\alpha \frac{\partial}{\partial \mu_\alpha}.$$

If $u^* \in E^*$ and $(u_x, v_{u^*}) \in E_x \times T_{u^*}E^*$ then

$$\theta_E(u^*)(u_x, v_{u^*}) = u^*(u_x),$$

is called the *Liouville section*. The *canonical symplectic section* ω_E is defined by

$$\omega_E = -d^E \theta_E,$$

and it results that is a nondegenerate 2-section and $d^E \omega_E = 0$.

On E^* we have the similar concept of the vertical lift to that in E . If $\alpha \in \Gamma(E^*)$ we can define the vector field α^v on E^* as follows

$$\alpha^v(u^*) = \alpha(\tau(u^*))_{u^*}^v, \quad u^* \in E^*,$$

where

$$_{u^*}^v : E_{\tau(u^*)}^* \rightarrow T_{u^*}(E_{\tau(u^*)}^*),$$

is the canonical isomorphism between the vector spaces $E_{\tau(u^*)}^*$ and $T_{u^*}(E_{\tau(u^*)}^*)$. Also, if X is a section of $\pi : E \rightarrow M$, there exists a unique vector field X^{*c} on E^* , called the complete lift of X to E^* , satisfying the two following conditions:

- (i) X^{*c} is τ -projectable on $\sigma(X)$,
- (ii) $X^{*c}(\hat{Y}) = \widehat{\mathcal{L}_X Y}$, for all $Y \in \Gamma(E)$ (see [41]).

If X is a section of E then \hat{X} is the linear function $\hat{X} \in C^\infty(E^*)$ given by

$$\hat{X}(u^*) = u^*(X(\tau(u^*))),$$

for all $u^* \in E^*$. Now, we may introduce the vertical lift α^v and the complete lift X^{*c} of a section $\alpha \in \Gamma(E^*)$ and a section $X \in \Gamma(E)$ as the sections of $\mathcal{T}E^*$ given by

$$\alpha^v(u^*) = (0, \alpha^v(u^*)), \quad X^{*c} = (X(\tau(u^*)), X^{*c}(u^*)), \quad u^* \in E^*.$$

The other canonical object on $\mathcal{T}E^*$ is *Liouville-Hamilton section* \mathcal{C} given by

$$\mathcal{C}(u^*) = (0, u_{u^*}^{*v}), \quad u^* \in E^*.$$

In local coordinates it follows that the Liouville section is given by

$$\theta_E = \mu_\alpha \mathcal{Q}^\alpha, \tag{1.5.5}$$

and we obtain

$$\omega_E = \mathcal{Q}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{Q}^\beta \wedge \mathcal{Q}^\gamma. \tag{1.5.6}$$

The *Liouville-Hamilton section* \mathcal{C} has local expression

$$\mathcal{C} = \mu_\alpha \mathcal{P}^\alpha. \tag{1.5.7}$$

If θ is a section of E^* , $\theta = \theta_\alpha s^\alpha$, and X is a section of E , $X = X^\alpha s_\alpha$ then the vertical and complete lifts have the expression

$$\theta^\vee = \theta_\alpha \mathcal{P}^\alpha,$$

$$X^{*c} = X^\alpha \mathcal{Q}_\alpha - \left(\sigma_\alpha^i \frac{\partial X^\varepsilon}{\partial q^i} + L_{\alpha\beta}^\varepsilon X^\beta \right) \mu_\varepsilon \mathcal{P}^\alpha.$$

The Liouville-Hamilton section on $\mathcal{T}E^*$ measures the homogeneity of the functions and sections. A function $f \in C^\infty(E^*)$ is said to be homogeneous of degree $r \in \mathbb{Z}$ if

$$\mathcal{L}_\mathcal{C} f = r f,$$

where $\mathcal{L}_\mathcal{C}$ is the Lie derivation with respect to the Liouville-Hamilton section on the Lie algebroid. A section ρ of $\mathcal{T}E^*$ is said to be homogeneous of degree $r \in \mathbb{Z}$ if

$$\mathcal{L}_\mathcal{C} \rho = r \rho.$$

We remark that $V\mathcal{T}E^*$ is Lagrangian for ω_E , i.e. $\omega_E(\rho_1, \rho_2) = 0$, for every vertical sections $\rho_1, \rho_2 \in \Gamma(V\mathcal{T}E^*)$.

1.5.1 Ehresmann connections on the Lie algebroid $\mathcal{T}E^*$

Definition 1.5.1 *The Ehresmann nonlinear connection on $\mathcal{T}E^*$ is an almost product structure \mathcal{N} on $\tau_1 : \mathcal{T}E^* \rightarrow E^*$ (i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E^* \rightarrow \mathcal{T}E^*$, such that $\mathcal{N}^2 = Id$) smooth on $\mathcal{T}E^* \setminus \{0\}$ such that $V\mathcal{T}E^* = \ker(Id + \mathcal{N})$.*

If \mathcal{N} is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^* = \ker(Id - \mathcal{N})$ is the horizontal distribution associated to \mathcal{N} and

$$\mathcal{T}E^* = V\mathcal{T}E^* \oplus H\mathcal{T}E^*.$$

Each $\rho \in \Gamma(\mathcal{T}E^*)$ can be written as $\rho = \rho^h + \rho^v$ where ρ^h, ρ^v are sections in the horizontal and respective, vertical subbundles. If $\rho^h = 0$ then ρ is called *vertical* and if $\rho^v = 0$ then ρ is called *horizontal*. A connection \mathcal{N} on E^* induces two projectors $h, v : \mathcal{T}E^* \rightarrow \mathcal{T}E^*$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in \Gamma(\mathcal{T}E^*)$. We have

$$h = \frac{1}{2}(Id + \mathcal{N}), \quad v = \frac{1}{2}(Id - \mathcal{N}),$$

$$\ker h = \text{Im} v = V\mathcal{T}E^*, \quad \text{Im} h = \ker v = H\mathcal{T}E^*,$$

$$h^2 = h, \quad v^2 = v, \quad hv = vh = 0, \quad h + v = Id.$$

Locally, a connection can be expressed as

$$\mathcal{N}(\mathcal{Q}_\alpha) = \mathcal{Q}_\alpha + 2\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta, \quad \mathcal{N}(\mathcal{P}^\alpha) = -\mathcal{P}^\alpha,$$

where $\mathcal{N}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(q, \mu)$ are the local coefficients of \mathcal{N} . The local coordinate expression of \mathcal{N} is

$$\mathcal{N} = \mathcal{Q}_\alpha \otimes \mathcal{Q}^\alpha - \mathcal{P}^\alpha \otimes \mathcal{P}_\alpha + 2\mathcal{N}_{\alpha\beta}\mathcal{P}^\alpha \otimes \mathcal{Q}^\beta.$$

The local sections \mathcal{P}^α , ($\alpha = \overline{1, m}$) define a local frame of $V\mathcal{T}E^*$, and the sections

$$\delta_\alpha^* = (\mathcal{Q}_\alpha)^h = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta, \tag{1.5.8}$$

generate a local frame of $H\mathcal{T}E^*$. The frame $\{\delta_\alpha^*, \mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *adapted* to the direct sum decomposition. The dual adapted basis is $\{\mathcal{Q}^\alpha, \delta\mathcal{P}_\alpha\}$ where

$$\delta\mathcal{P}_\alpha = \mathcal{P}_\alpha - \mathcal{N}_{\alpha\beta}\mathcal{Q}^\beta. \tag{1.5.9}$$

It results that in the adapted basis the expression of Ehresmann connection becomes

$$\mathcal{N} = \delta_\alpha^* \otimes \mathcal{Q}^\alpha + \mathcal{P}^\alpha \otimes \delta\mathcal{P}_\alpha,$$

Definition 1.5.2 A connection \mathcal{N} is called symmetric if $HT E^*$ is Lagrangian for ω_E .

By a straightforward computation, using (1.5.8) we get

$$\omega_E(\delta_\alpha^*, \delta_\beta^*) = \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} - \mu_\gamma L_{\alpha\beta}^\gamma,$$

and it results that \mathcal{N} is symmetric if and only if

$$\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L_{\alpha\beta}^\gamma. \quad (1.5.10)$$

Proposition 1.5.1 With respect to a symmetric nonlinear connection, the canonical symplectic structure ω_E can be written in the following form

$$\omega_E = \mathcal{Q}^\alpha \wedge \delta \mathcal{P}_\alpha + \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{Q}^\beta \wedge \mathcal{Q}^\gamma. \quad (1.5.11)$$

Proof. Using (1.5.6) and (1.5.9) we get

$$\omega_E = \mathcal{Q}^\alpha \wedge \delta \mathcal{P}_\alpha + \frac{1}{2}(\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha}) \mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{Q}^\beta \wedge \mathcal{Q}^\gamma,$$

which ends the proof. \square

Proposition 1.5.2 The Lie brackets of the adapted basis $\{\delta_\alpha^*, \mathcal{P}^\alpha\}$ are [50]

$$[\delta_\alpha^*, \delta_\beta^*]_{TE^*} = L_{\alpha\beta}^\gamma \delta_\gamma^* + \mathcal{R}_{\alpha\beta\gamma} \mathcal{P}^\gamma, \quad [\delta_\alpha^*, \mathcal{P}^\beta]_{TE^*} = -\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\beta} \mathcal{P}^\gamma, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{TE^*} = 0,$$

$$\mathcal{R}_{\alpha\beta\gamma} = \sigma_\alpha^i \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial q^i} - \sigma_\beta^i \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial q^i} + \mathcal{N}_{\alpha\delta} \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\delta} - \mathcal{N}_{\beta\delta} \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\delta} - L_{\alpha\beta}^\varepsilon \mathcal{N}_{\varepsilon\gamma}. \quad (1.5.12)$$

Proof. Using (1.5.8) we obtain

$$[\delta_\alpha^*, \delta_\beta^*]_{TE^*} = \left(\sigma_\alpha^i \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial q^i} - \sigma_\beta^i \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial q^i} + \mathcal{N}_{\alpha\delta} \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\delta} - \mathcal{N}_{\beta\delta} \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\delta} \right) \mathcal{P}^\gamma + L_{\alpha\beta}^\varepsilon \mathcal{Q}_\varepsilon,$$

and putting $\mathcal{Q}_\varepsilon = \delta_\varepsilon^* - \mathcal{N}_{\varepsilon\gamma} \mathcal{P}^\gamma$ we get $[\delta_\alpha^*, \delta_\beta^*]_{TE^*} = L_{\alpha\beta}^\gamma \delta_\gamma^* + \mathcal{R}_{\alpha\beta\gamma} \mathcal{P}^\gamma$. \square

The curvature of a connection \mathcal{N} on TE^* is given by $\Omega = -\mathbf{N}_h$, where h is the horizontal projector and \mathbf{N}_h is the Nijenhuis tensor of h , given by

$$\mathbf{N}_h(\theta, \rho) = [h\theta, h\rho]_{TE^*} - h[h\theta, \rho]_{TE^*} - h[\theta, h\rho]_{TE^*} + h^2[\theta, \rho]_{TE^*}.$$

Proposition 1.5.3 In local coordinates we get

$$\Omega = -\frac{1}{2} \mathcal{R}_{\alpha\beta\gamma} \mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta \otimes \mathcal{P}^\gamma, \quad (1.5.13)$$

where $\mathcal{R}_{\alpha\beta\gamma}$ is given by (1.5.12) and are called the coefficients of the curvature tensor of \mathcal{N} .

Proof. Since $h^2 = h$ we obtain

$$\Omega(h\rho_1, h\rho_2) = -v[h\rho_1, h\rho_2]_{\mathcal{T}E^*}, \quad \Omega(h\rho_1, v\rho_2) = \Omega(v\rho_1, v\rho_2) = 0,$$

and in local coordinates we get

$$\Omega(\delta_\alpha^*, \delta_\beta^*) = -v[\delta_\alpha^*, \delta_\beta^*]_{\mathcal{T}E^*} = -\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma,$$

which concludes the proof. \square

Proposition 1.5.4 *The curvature satisfies the Bianchi identity*

$$\mathcal{R}_{\alpha\beta\gamma} + \mathcal{R}_{\beta\gamma\alpha} + \mathcal{R}_{\gamma\alpha\beta} = 0.$$

Proof. By direct computation, using the relation (1.5.12) and structure equations given by (1.2.9), (1.2.10).

The curvature is an obstruction to the integrability of $H\mathcal{T}E^*$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid bracket of $\mathcal{T}E^*$. We have:

Remark 1.5.1 *$H\mathcal{T}E^*$ is integrable if and only if the curvature vanishes.*

Also, the integrability conditions for the almost product structure \mathcal{N} is given by the vanishing of the associated Nijenhuis tensor $N_{\mathcal{N}}$. By a straightforward computation we obtain

$$N_{\mathcal{N}}(\mathcal{P}^\alpha, \mathcal{P}^\beta) = 0, \quad N_{\mathcal{N}}(\delta_\alpha^*, \mathcal{P}^\beta) = 0, \quad N_{\mathcal{N}}(\delta_\alpha^*, \delta_\beta^*) = 4\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma.$$

Thus

$$N_{\mathcal{N}} = -2\mathcal{R}_{\alpha\beta\gamma}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma,$$

and it results that the distribution $H\mathcal{T}E^*$ is integrable if and only if the almost product structure \mathcal{N} is integrable.

We consider the connections \mathcal{N} on $\mathcal{T}E^*$ and N on TE^* which are σ^1 -related and the adapted basis $(\delta_i, \frac{\partial}{\partial \mu_\alpha})$ of N given by $\delta_i = \frac{\partial}{\partial q^i} + N_{i\alpha}\frac{\partial}{\partial \mu_\alpha}$, where $N_{i\beta}$ are the coefficients of N .

Theorem 1.5.1 *The following relations hold*

$$\sigma^1(\delta_\alpha^*) = \sigma_\alpha^i \delta_i, \quad \mathcal{N}_{\alpha\beta} = \sigma_\alpha^i N_{i\beta}, \quad \mathcal{R}_{\alpha\beta\gamma} = \sigma_\alpha^i \sigma_\beta^j R_{ij\gamma}, \quad (1.5.14)$$

where

$$R_{ij\gamma} = \delta_i(N_{j\gamma}) - \delta_j(N_{i\gamma}),$$

is the curvature of the nonlinear connection N on TE^* .

Proof. Since $\mathcal{N}(\sigma^1(\delta_\alpha^*)) = \sigma^1(\delta_\alpha^*)$ the relation $N \circ \sigma^1 = \sigma^1 \circ \mathcal{N}$ leads to $N(\sigma^1(\delta_\alpha^*)) = \sigma^1(\delta_\alpha^*)$. But $N(\sigma^1(\delta_\alpha^*)) = 2\sigma_\alpha^i \delta_i - \sigma^1(\delta_\alpha^*)$ which concludes that $\sigma^1(\delta_\alpha^*) = \sigma_\alpha^i \delta_i$, from which we easily obtain $\mathcal{N}_{\alpha\beta} = \sigma_\alpha^i N_{i\beta}$. By straightforward computation we get

$$\mathcal{R}_{\alpha\beta\gamma} = \sigma_\alpha^i \sigma_\beta^j (\delta_i(N_{j\gamma}) - \delta_j(N_{i\gamma})) + N_{j\gamma} \left(\sigma_\beta^i \frac{\partial \sigma_\alpha^j}{\partial q^i} - \sigma_\alpha^i \frac{\partial \sigma_\beta^j}{\partial q^i} \right) + L_{\alpha\beta}^\varepsilon \mathcal{N}_{\varepsilon\gamma},$$

and using (1.2.10), the second term is given by $N_{j\gamma} \sigma_\beta^j L_{\beta\alpha}^\varepsilon = -N_{\varepsilon\gamma} L_{\alpha\beta}^\varepsilon$, which ends the proof. \square

Remark 1.5.2 A σ^1 -related connection N on TE^* determines a connection \mathcal{N} on \mathcal{TE}^* with the coefficients given by

$$\mathcal{N}_{\alpha\beta} = \sigma_\alpha^i N_{i\beta},$$

and curvature

$$\mathcal{R}_{\alpha\beta\gamma} = \sigma_\alpha^i \sigma_\beta^j R_{ij\gamma}.$$

Conversely, it is not true, because σ is only injective.

1.5.2 Regular sections and connections

Definition 1.5.3 *An almost tangent structure \mathcal{J} on $\mathcal{T}E^*$ is a bundle morphism $\mathcal{J} : \mathcal{T}E^* \rightarrow \mathcal{T}E^*$ of $\tau_1 : \mathcal{T}E^* \rightarrow E^*$ of rank m , such that $\mathcal{J}^2 = 0$. An almost tangent structure \mathcal{J} on $\mathcal{T}E^*$ is called adapted if*

$$\text{Im}\mathcal{J} = \ker \mathcal{J} = V\mathcal{T}E^*.$$

Locally, an adapted almost tangent structure is given by

$$\mathcal{J} = t_{\alpha\beta} \mathcal{Q}^\alpha \otimes \mathcal{P}^\beta, \quad (1.5.15)$$

where the tensor $t_{\alpha\beta}(x, \mu)$ is nondegenerate.

Now, we find the integrability conditions for the adapted almost tangent structure. From [50] it results

Proposition 1.5.5 *\mathcal{J} is an integrable structure if and only if*

$$\frac{\partial t^{\alpha\gamma}}{\partial \mu_\beta} = \frac{\partial t^{\beta\gamma}}{\partial \mu_\alpha}, \quad (1.5.16)$$

where $t^{\alpha\gamma} t_{\gamma\beta} = \delta_\beta^\alpha$.

Proof. \mathcal{J} is integrable if and only if the associated Nijenhuis tensor

$$\text{N}_{\mathcal{J}}(\rho, \nu) = [\mathcal{J}\rho, \mathcal{J}\nu]_{\mathcal{T}E^*} - \mathcal{J}[\mathcal{J}\rho, \nu]_{\mathcal{T}E^*} - \mathcal{J}[\rho, \mathcal{J}\nu]_{\mathcal{T}E^*},$$

vanishes. This is locally equivalent with the relations

$$\text{N}_{\mathcal{J}}(\mathcal{X}_\alpha, \mathcal{X}_\beta) = \left(t_{\alpha\gamma} \frac{\partial t_{\beta\varepsilon}}{\partial \mu_\gamma} - t_{\beta\gamma} \frac{\partial t_{\alpha\varepsilon}}{\partial \mu_\gamma} \right) \mathcal{P}^\varepsilon, \quad \text{N}_{\mathcal{J}}(\mathcal{X}_\alpha, \mathcal{P}^\beta) = \text{N}_{\mathcal{J}}(\mathcal{P}^\alpha, \mathcal{P}^\beta) = 0.$$

Therefore \mathcal{J} is integrable if and only if

$$t_{\alpha\gamma} \frac{\partial t_{\beta\varepsilon}}{\partial \mu_\gamma} = t_{\beta\gamma} \frac{\partial t_{\alpha\varepsilon}}{\partial \mu_\gamma},$$

that is equivalent to (1.5.16).

Definition 1.5.4 *An adapted almost tangent structure \mathcal{J} on $\mathcal{T}E^*$ is called symmetric if*

$$\omega_E(\mathcal{J}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_2, \rho_1). \quad (1.5.17)$$

Locally, this requires the symmetry of the tensor $t_{\alpha\beta}$.

Remark 1.5.3 *If g is a pseudo-Riemannian metric on the vertical bundle $V\mathcal{T}E^*$ then there exists a unique symmetric adapted almost tangent structure on $\mathcal{T}E^*$ such that*

$$g(\mathcal{J}\rho, \mathcal{J}v) = -\omega_E(\mathcal{J}\rho, v), \quad (1.5.18)$$

and we say that \mathcal{J} is induced by the metric g .

Locally, if

$$g(q, \mu) = g^{\alpha\beta} \mathcal{P}_\alpha \otimes \mathcal{P}_\beta,$$

then the relation (1.5.18) implies $t^{\alpha\beta} = g^{\alpha\beta}$.

Remark 1.5.4 *Any symmetric adapted almost tangent structure \mathcal{J} on $\mathcal{T}E^*$ induces a pseudo-Riemannian metric on the vertical bundle $V\mathcal{T}E^*$ as defined by (1.5.18).*

Definition 1.5.5 *The torsion of a connection \mathcal{N} is the vector valued two form $T = [\mathcal{J}, h]$, where h is the horizontal projector and $[\mathcal{J}, h]$ is the Frölicher-Nijenhuis bracket*

$$\begin{aligned} [\mathcal{J}, h](X, Y) &= [\mathcal{J}X, hY]_{\mathcal{T}E^*} + [hX, \mathcal{J}Y]_{\mathcal{T}E^*} + \mathcal{J}[X, Y]_{\mathcal{T}E^*} - \mathcal{J}[X, hY]_{\mathcal{T}E^*} - \\ &\quad - \mathcal{J}[hX, Y]_{\mathcal{T}E^*} - h[X, \mathcal{J}Y]_{\mathcal{T}E^*} - h[\mathcal{J}X, Y]_{\mathcal{T}E^*}, \end{aligned}$$

The torsion T is a semibasic vector-valued form. Its local expression is

$$T = \frac{1}{2} T_{\alpha\beta\gamma} \mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta \otimes \mathcal{P}^\gamma,$$

where

$$T_{\alpha\beta\gamma} = t_{\alpha\epsilon} \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\epsilon} - t_{\beta\epsilon} \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\epsilon} + \delta_\alpha^*(t_{\beta\gamma}) - \delta_\beta^*(t_{\alpha\gamma}) - L_{\alpha\beta}^\epsilon t_{\epsilon\gamma}. \quad (1.5.19)$$

Next, let us consider the linear mapping $\mathcal{F} : \mathcal{T}E^* \rightarrow \mathcal{T}E^*$ given by

$$\mathcal{F}(h\rho) = \mathcal{J}\rho, \quad \mathcal{F}(\mathcal{J}\rho) = -h\rho, \quad (1.5.20)$$

where $\rho \in \Gamma(\mathcal{T}E^*)$, and h is the horizontal projector induced by the nonlinear connection.

Proposition 1.5.6 *The mapping \mathcal{F} has the properties:*

- a) \mathcal{F} is an almost complex structure $\mathcal{F} \circ \mathcal{F} = -Id$.
- b) Locally it is given by

$$\mathcal{F} = t_{\alpha\beta} \mathcal{P}^\beta \otimes \mathcal{Q}^\alpha - t^{\alpha\beta} \delta_\alpha^* \otimes \delta \mathcal{P}_\beta. \quad (1.5.21)$$

Proof. It results by definition that

$$(\mathcal{F} \circ \mathcal{F})(h\rho) = \mathcal{F}(\mathcal{J}(\rho)) = -h\rho, \quad (\mathcal{F} \circ \mathcal{F})(\mathcal{J}\rho) = \mathcal{F}(-h\rho) = -\mathcal{J}\rho,$$

$$\mathcal{F}(\delta_\alpha^*) = t_{\alpha\beta} \mathcal{P}^\beta, \quad \mathcal{F}(\mathcal{P}^\alpha) = -t^{\alpha\beta} \delta_\beta^*,$$

which concludes the proof. \square

Proposition 1.5.7 *The almost complex structure is integrable if and only if the torsion and curvature of the connection satisfy the equations*

$$T_{\alpha\beta\gamma} = 0, \quad \mathcal{R}_{\alpha\beta\gamma} = t_{\alpha\varepsilon} \frac{\partial t_{\beta\gamma}}{\partial \mu_\varepsilon} - t_{\beta\varepsilon} \frac{\partial t_{\alpha\gamma}}{\partial \mu_\varepsilon}. \quad (1.5.22)$$

Proof. Let $N_{\mathcal{F}}$ be the Nijenhuis tensor of the almost complex structure. We get

$$\begin{aligned} N_{\mathcal{F}}(\delta_\alpha^*, \delta_\beta^*) &= N_{\alpha\beta}^\gamma \delta_\gamma^* + N_{\alpha\beta(\gamma)} \mathcal{P}^\gamma, \\ N_{\mathcal{F}}(\delta_\alpha^*, \mathcal{P}^\beta) &= N_\alpha^{(\beta)\gamma} \delta_\gamma^* + N_{\alpha(\gamma)}^{(\beta)} \mathcal{P}^\gamma, \\ N_{\mathcal{F}}(\mathcal{P}^\alpha, \mathcal{P}^\beta) &= -t^{\varepsilon\alpha} t^{\beta\tau} N_{\mathcal{F}}(\delta_\varepsilon^*, \delta_\tau^*), \end{aligned}$$

where

$$\begin{aligned} N_{\alpha\beta}^\gamma &= T_{\alpha\beta\varepsilon} t^{\varepsilon\gamma}, \quad N_{\alpha\beta(\gamma)} = t_{\alpha\varepsilon} \frac{\partial t_{\beta\gamma}}{\partial \mu_\varepsilon} - t_{\beta\varepsilon} \frac{\partial t_{\alpha\gamma}}{\partial \mu_\varepsilon} - \mathcal{R}_{\alpha\beta\gamma}, \\ N_{\alpha\beta(\gamma)} &= N_\alpha^{(\varepsilon)\tau} t_{\varepsilon\beta} t_{\gamma\tau}, \quad N_{\alpha\beta}^\gamma = N_{\alpha(\tau)}^{(\varepsilon)} t^{\tau\gamma} t_{\beta\varepsilon}, \end{aligned}$$

which ends the proof. \square

Remark 1.5.5 *Let \mathcal{N} be a bundle morphism of $\tau_1 : \mathcal{T}E^* \rightarrow E^*$, smooth on $\mathcal{T}E^* \setminus \{0\}$. Then \mathcal{N} is a connection on $\mathcal{T}E^*$ if and only if there exists an adapted almost tangent structure \mathcal{J} on $\mathcal{T}E^*$ such that*

$$\mathcal{J}\mathcal{N} = \mathcal{J}, \quad \mathcal{N}\mathcal{J} = -\mathcal{J}. \quad (1.5.23)$$

Definition 1.5.6 *Let \mathcal{J} be an adapted tangent structure on $\mathcal{T}E^*$. A section ρ of $\mathcal{T}E^*$ is called \mathcal{J} -regular if*

$$\mathcal{J}[\rho, \mathcal{J}\nu]_{\mathcal{T}E^*} = -\mathcal{J}\nu, \quad (1.5.24)$$

for every section ν of $\mathcal{T}E^*$.

Locally, the section $\rho = \xi^\alpha \mathcal{Q}_\alpha + \rho_\beta \mathcal{P}^\beta$ is \mathcal{J} -regular if and only if

$$t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha}.$$

where $t^{\alpha\beta} t_{\alpha\gamma} = \delta_\gamma^\beta$.

We have to remark that if the equation (1.5.24) is satisfied for any section $\nu \in \Gamma(\mathcal{T}E^*)$ with $\text{rank}[t^{\alpha\beta}] = m$, then \mathcal{J} is an *integrable* structure. Indeed, we have

$$\frac{\partial t^{\alpha\beta}}{\partial \mu_\gamma} = \frac{\partial^2 \xi^\beta}{\partial \mu_\gamma \partial \mu_\alpha} = \frac{\partial^2 \xi^\beta}{\partial \mu_\alpha \partial \mu_\gamma} = \frac{\partial t^{\gamma\beta}}{\partial \mu_\alpha},$$

and using (1.5.16) it follows that \mathcal{J} is integrable.

From [50] we have:

Theorem 1.5.2 *Let \mathcal{J} be an adapted tangent structure on $\mathcal{T}E^*$. If ρ is a \mathcal{J} -regular section of $\mathcal{T}E^*$ then*

$$\mathcal{N} = -\mathcal{L}_\rho \mathcal{J}, \quad (1.5.25)$$

is a connection on $\mathcal{T}E^$.*

Proof. Since

$$\mathcal{N}(v) = -\mathcal{L}_\rho \mathcal{J}(v) = -[\rho, \mathcal{J}v]_{\mathcal{T}E^*} + \mathcal{J}[\rho, v]_{\mathcal{T}E^*},$$

then

$$\mathcal{J}\mathcal{N}(v) = -\mathcal{J}[\rho, \mathcal{J}v]_{\mathcal{T}E^*} + \mathcal{J}^2[\rho, v]_{\mathcal{T}E^*} = \mathcal{J}v,$$

$$\mathcal{N}\mathcal{J}(v) = -[\rho, \mathcal{J}^2v]_{\mathcal{T}E^*} + \mathcal{J}[\rho, \mathcal{J}v]_{\mathcal{T}E^*} = -\mathcal{J}v.$$

and using (1.5.23) it results the conclusion. \square

This connection is induced by \mathcal{J} and ρ . Its local coefficients are given by

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2} \left(t_{\alpha\gamma} \frac{\partial \rho_\beta}{\partial \mu_\gamma} - \sigma_\alpha^i t_{\gamma\beta} \frac{\partial \xi^\gamma}{\partial q^i} - \rho(t_{\alpha\beta}) + \xi^\gamma t_{\varepsilon\beta} L_{\gamma\alpha}^\varepsilon \right). \quad (1.5.26)$$

Definition 1.5.7 *An adapted tangent structure \mathcal{J} on $\mathcal{T}E^*$ is called homogeneous if*

$$\mathcal{L}_\mathcal{C} \mathcal{J} = -\mathcal{J}.$$

Notice that \mathcal{J} is homogeneous if the local components $t_{\alpha\beta}(x, \mu)$ are 0-homogeneous with respect to μ .

Proposition 1.5.8 *Let \mathcal{J} be a homogeneous adapted tangent structure. A section ρ of $\mathcal{T}E^*$ is \mathcal{J} -regular if and only if*

$$\mathcal{J}\rho = \mathcal{C}.$$

Proof. If ρ is \mathcal{J} -regular then

$$t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha},$$

is 0-homogeneous, hence ξ^β must be 1-homogeneous with respect to μ , therefore $\xi^\beta = \mu_\alpha t^{\alpha\beta}$, that is equivalent to $\mathcal{J}\rho = \mathcal{C}$. Vice versa, if $\mathcal{J}\rho = \mathcal{C}$ then $\xi^\beta = \mu_\alpha t^{\alpha\beta}$ and thus

$$\frac{\partial \xi^\beta}{\partial \mu_\gamma} = t^{\gamma\beta} + \mu_\varepsilon \frac{t^{\gamma\beta}}{\partial \mu_\varepsilon} = t^{\gamma\beta},$$

which ends the proof. \square

Remark 1.5.6 (i) Based on the above result, the local expression for a \mathcal{J} -regular section with \mathcal{J} a homogeneous adapted tangent structure is

$$\rho = \mu_\alpha t^{\alpha\beta} \mathcal{X}_\beta + \rho_\alpha \mathcal{P}^\alpha. \quad (1.5.27)$$

(ii) The coefficients (1.5.26) generated by ρ from (1.5.27) can be written in the following form

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2} \left(\left(\sigma_\alpha^i \frac{\partial t_{\gamma\beta}}{\partial x^i} - \sigma_\gamma^i \frac{\partial t_{\alpha\beta}}{\partial x^i} \right) t^{\varepsilon\gamma} \mu_\varepsilon + t_{\alpha\varepsilon} \frac{\partial \rho_\beta}{\partial \mu_\varepsilon} - \rho_\varepsilon \frac{\partial t_{\alpha\beta}}{\partial \mu_\varepsilon} + \mu_\varepsilon t^{\varepsilon\gamma} L_{\gamma\alpha}^\lambda t_{\lambda\beta} \right).$$

1.5.3 Hamilton sections

Definition 1.5.8 A section ψ on $\mathcal{T}E^*$ is called a Hamilton section if it is \mathcal{J} -regular and

$$\mathcal{L}_\psi \omega_E = 0,$$

where ω_E is the canonical symplectic section.

If $\psi = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ then the condition $\mathcal{L}_\psi \omega_E = 0$ is expressed locally by [109]

$$\begin{aligned} i) \quad & \frac{\partial \xi^\beta}{\partial \mu_\alpha} = \frac{\partial \xi^\alpha}{\partial \mu_\beta}. \\ ii) \quad & \sigma_\alpha^i \frac{\partial \xi^\beta}{\partial q^i} + \frac{\partial \rho_\alpha}{\partial \mu_\beta} = \xi^\gamma L_{\gamma\alpha}^\beta + \mu_\varepsilon L_{\gamma\alpha}^\varepsilon \frac{\partial \xi^\gamma}{\partial \mu_\beta}. \end{aligned} \quad (1.5.28)$$

$$\begin{aligned} iii) \quad & \sigma_\beta^i \frac{\partial \rho_\alpha}{\partial q^i} - \sigma_\alpha^i \frac{\partial \rho_\beta}{\partial q^i} = \mu_\varepsilon \xi^\gamma \left(\sigma_\beta^i \frac{\partial L_{\gamma\alpha}^\varepsilon}{\partial q^i} - \sigma_\alpha^i \frac{\partial L_{\gamma\beta}^\varepsilon}{\partial q^i} + L_{\nu\gamma}^\varepsilon L_{\alpha\beta}^\nu \right) + \\ & + \mu_\varepsilon \frac{\partial \xi^\gamma}{\partial q^i} (\sigma_\beta^i L_{\gamma\alpha}^\varepsilon - \sigma_\alpha^i L_{\gamma\beta}^\varepsilon) - \rho_\gamma L_{\alpha\beta}^\gamma. \end{aligned}$$

Now, we deal with some generalizations of the Hamilton sections.

Definition 1.5.9 The section $\psi = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ on $\mathcal{T}E^*$ defines a mechanical structure if ψ is \mathcal{J} -regular and

$$\omega_E(\mathcal{J}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_2, \rho_1), \quad (1.5.29)$$

for any $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$.

The definition is equivalent with the symmetry of $t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha}$, which means that the property (1.5.28) i) is fulfilled.

Proposition 1.5.9 The section ψ on $\mathcal{T}E^*$ defines a mechanical structure if and only if

$$\mathcal{L}_\psi \omega_E(\nu_1, \nu_2) = 0,$$

whenever $\nu_1, \nu_2 \in \Gamma(V\mathcal{T}E^*)$.

Proof. Let us consider $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$ and $\nu_1 = \mathcal{J}\rho_1, \nu_2 = \mathcal{J}\rho_2$. Then

$$\begin{aligned} \mathcal{L}_\psi \omega_E(\mathcal{J}\rho_1, \mathcal{J}\rho_2) &= \psi(\omega_E(\mathcal{J}\rho_1, \mathcal{J}\rho_2) - \omega_E((\mathcal{L}_\psi \mathcal{J}\rho_1, \mathcal{J}\rho_2) - \\ - \omega_E((\mathcal{J}\rho_1, \mathcal{L}_\psi \mathcal{J}\rho_2) &= \omega_E((\mathcal{N}\rho_1, \mathcal{J}\rho_2) + \omega_E((\mathcal{J}\rho_1, \mathcal{N}\rho_2)). \end{aligned}$$

Considering $\mathcal{N}\rho_1, \mathcal{N}\rho_2$ for ρ_1, ρ_2 in the previous relation and using the properties (1.5.23) and $\mathcal{N}^2 = Id$ we obtain

$$\mathcal{L}_\psi \omega_E(\mathcal{J}\rho_1, \mathcal{J}\rho_2) = \mathcal{L}_\psi \omega_E(\mathcal{JN}\rho_1, \mathcal{JN}\rho_2) = \omega_E(\rho_1, \mathcal{J}\rho_2) + \omega_E(\mathcal{J}\rho_1, \rho_2),$$

which ends the proof. \square

Definition 1.5.10 *The section ψ on $\mathcal{T}E^*$ is a semi-Hamilton section if ψ is \mathcal{J} -regular and*

$$i_v(\mathcal{L}_\psi \omega_E) = 0, \quad (1.5.30)$$

whenever $v \in \Gamma(V\mathcal{T}E^)$.*

In the case of a semi-Hamilton section on $\mathcal{T}E^*$ only the conditions (1.5.28), *i*) and *ii*) are satisfied.

Let us consider a section ψ defining a mechanical structure on $\mathcal{T}E^*$, then we can find the other connection. Indeed, we consider

$$g(\rho_1, \rho_2) = -\omega_E(\mathcal{J}\rho_1, \rho_2),$$

for $\rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$. It results that $g(\rho_1, \rho_2) = g(\rho_2, \rho_1)$ and $g(v, \rho) = 0$ whenever $v \in \Gamma(V\mathcal{T}E^*)$. The local coordinate expression of g is given by

$$g = g_{\alpha\beta} \mathcal{Q}^\alpha \otimes \mathcal{Q}^\beta.$$

Proposition 1.5.10 *If ψ defines a mechanical structure on $\mathcal{T}E^*$ then the section \mathcal{N}' defined by*

$$\omega_E(\mathcal{N}'\rho_1, \rho_2) = (\mathcal{L}_\psi g)(\rho_1, \rho_2), \quad \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*), \quad (1.5.31)$$

determines a connection on $\mathcal{T}E^$.*

Proof. First, we show that the following relation holds

$$(\mathcal{L}_\psi g)(\rho_1, \rho_2) = -(\mathcal{L}_\psi \omega_E)(\mathcal{J}\rho_1, \rho_2) + \omega_E(\mathcal{N}\rho_1, \rho_2).$$

Indeed, we get

$$\begin{aligned} (\mathcal{L}_\psi g)(\rho_1, \rho_2) &= \psi(g(\rho_1, \rho_2)) - g(\mathcal{L}_\psi \rho_1, \rho_2) - g(\rho_1, \mathcal{L}_\psi \rho_2) = \\ &= -\psi(\omega_E(\mathcal{J}\rho_1, \rho_2)) + \omega_E(\mathcal{J}(\mathcal{L}_\psi \rho_1), \rho_2) + \omega_E(\mathcal{J}\rho_1, \mathcal{L}_\psi \rho_2). \end{aligned}$$

But the relation

$$\psi(\omega_E(\mathcal{J}\rho_1, \rho_2)) = (\mathcal{L}_\psi \omega_E)(\mathcal{J}\rho_1, \rho_2) + \omega_E(\mathcal{L}_\psi(\mathcal{J}\rho_1), \rho_2) + \omega_E(\mathcal{J}\rho_1, \mathcal{L}_\psi \rho_2),$$

yields

$$(\mathcal{L}_\psi g)(\rho_1, \rho_2) = -(\mathcal{L}_\psi \omega_E)(\mathcal{J}\rho_1, \rho_2) + \omega_E(-(\mathcal{L}_\psi \mathcal{J})\rho_1, \rho_2).$$

Also, it results

$$(\mathcal{L}_\psi \omega_E)(\mathcal{J}\rho_1, \rho_2) = \omega_E((\mathcal{N} - \mathcal{N}')\rho_1, \rho_2). \quad (1.5.32)$$

But,

$$\begin{aligned} \omega_E(\mathcal{N}'\mathcal{J}\rho_1, \rho_2) &= (\mathcal{L}_\psi g)(\mathcal{J}\rho_1, \rho_2) = \\ &= \psi(g(\mathcal{J}\rho_1, \rho_2)) - g((\mathcal{L}_\psi \mathcal{J})\rho_1, \rho_2) - g(\mathcal{J}\rho_1, \mathcal{L}_\psi \rho_2) = \\ &= g(\mathcal{N}\rho_1, \rho_2) = -\omega_E(\mathcal{J}\mathcal{N}\rho_1, \rho_2) = -\omega_E(\mathcal{J}\rho_1, \rho_2), \end{aligned}$$

whence $\mathcal{N}'\mathcal{J} = -\mathcal{J}$. Next

$$\omega_E(\mathcal{J}\mathcal{N}'\rho_1, \rho_2) = -\omega_E(\mathcal{N}'\rho_1, \mathcal{J}\rho_2) = -(\mathcal{L}_\psi g)(\rho_1, \mathcal{J}\rho_2) =$$

$$(\mathcal{L}_\psi \omega_E)(\mathcal{J}\rho_1, \mathcal{J}\rho_2) - \omega_E(\mathcal{N}\rho_1, \mathcal{J}\rho_2) = \omega_E(\mathcal{J}\mathcal{N}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_1, \rho_2),$$

whence $\mathcal{J}\mathcal{N}' = \mathcal{J}$, which ends the proof. \square

In local coordinates the connection \mathcal{N}' has the coefficients given by

$$\mathcal{N}'_{\alpha\beta} = \frac{1}{2} \left(-\psi t_{\alpha\beta} - \sigma_\beta^i \frac{\partial \xi^\varepsilon}{\partial q^i} t_{\varepsilon\alpha} - \sigma_\alpha^i \frac{\partial \xi^\varepsilon}{\partial q^i} t_{\varepsilon\beta} + \xi^\gamma L_{\gamma\beta}^\varepsilon t_{\varepsilon\alpha} + \xi^\gamma L_{\gamma\alpha}^\varepsilon t_{\varepsilon\beta} + \mu_\varepsilon L_{\beta\alpha}^\varepsilon \right).$$

Using (1.5.32) we obtain

Remark 1.5.7 *In the case that ψ is both semi-Hamiltonian and mechanical sections and moreover Hamiltonian, then the connections \mathcal{N} and \mathcal{N}' coincide.*

1.5.4 Hamiltonian formalism on Lie algebroids

Let us consider a differentiable and regular Hamiltonian $\mathcal{H} : E^* \rightarrow \mathbb{R}$ i.e. the matrix

$$g^{\alpha\beta}(q, \mu) = \frac{\partial^2 \mathcal{H}}{\partial \mu_\alpha \partial \mu_\beta},$$

is nondegenerate.

Any regular Hamiltonian \mathcal{H} on E^* induces a pseudo-Riemannian metric on $V\mathcal{T}E^*$ (the metric tensor is $g^{\alpha\beta}(q, \mu)$) therefore, it induces a unique symmetric adapted almost tangent structure (denoted $\mathcal{J}_{\mathcal{H}}$) such that (1.5.18) is verified. Moreover, this is a tangent structure i.e., $\mathcal{J}_{\mathcal{H}}$ is integrable.

A \mathcal{J} -regular section induced by the regular Hamiltonian \mathcal{H} is

$$\rho = \frac{\partial \mathcal{H}}{\partial \mu_\alpha} \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha.$$

Since ω_E is a symplectic section on the Lie algebroid $(\mathcal{T}E^*, [\cdot, \cdot]_{\mathcal{T}E^*}, \sigma^1)$ and $d\mathcal{H} \in \Gamma(\mathcal{T}E^*)$, we get

Remark 1.5.8 *There exists a unique section $\rho_{\mathcal{H}} \in \Gamma(\mathcal{T}E^*)$ such that*

$$i_{\rho_{\mathcal{H}}} \omega_E = d^E \mathcal{H},$$

and $\rho_{\mathcal{H}}$ is a *Hamilton section*, i.e. the relations (1.5.28) are fulfilled.

With respect to the local basis $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$, the local expression of $\rho_{\mathcal{H}}$ is

$$\rho_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial \mu_\alpha} \mathcal{Q}_\alpha - \left(\sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial q^i} + \mu_\gamma L_{\alpha\beta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\beta} \right) \mathcal{P}^\alpha, \quad (1.5.33)$$

Thus, the vector field $\sigma^1(\rho_{\mathcal{H}})$ on E^* is given by [64]

$$\sigma^1(\rho_{\mathcal{H}}) = \sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial \mu_\alpha} \frac{\partial}{\partial q^i} - \left(\sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial q^i} + \mu_\gamma L_{\alpha\beta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\beta} \right) \frac{\partial}{\partial \mu_\alpha},$$

and consequently, the integral curves of $\rho_{\mathcal{H}}$ (i.e. the integral curves of the vector field $\sigma^1(\rho_{\mathcal{H}})$) satisfy the Hamilton equations

$$\frac{dq^i}{dt} = \sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial q^i} - \mu_\gamma L_{\alpha\beta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\beta}. \quad (1.5.34)$$

The *Theorem 1.5.2* yields [103]

Corollary 1.5.1 *The symmetric nonlinear connection $\mathcal{N} = -\mathcal{L}_{\rho_{\mathcal{H}}} \mathcal{J}_{\mathcal{H}}$ has the coefficients given by*

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2}(\sigma_\gamma^i \{g_{\alpha\beta}, \mathcal{H}\} - \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_\varepsilon} (\sigma_\beta^i g_{\alpha\varepsilon} + \sigma_\alpha^i g_{\beta\varepsilon})) +$$

$$+ \mu_\gamma L_{\varepsilon\kappa}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\varepsilon} \frac{\partial g_{\alpha\beta}}{\partial \mu_\kappa} + \mu_\gamma L_{\alpha\beta}^\gamma + \frac{\partial \mathcal{H}}{\partial \mu_\delta} (g_{\alpha\varepsilon} L_{\delta\beta}^\varepsilon + g_{\beta\varepsilon} L_{\delta\alpha}^\varepsilon), \quad (1.5.35)$$

where

$$\{g_{\alpha\beta}, \mathcal{H}\} = \frac{\partial g_{\alpha\beta}}{\partial \mu_\gamma} \frac{\partial \mathcal{H}}{\partial q^i} - \frac{\partial g_{\alpha\beta}}{\partial q^i} \frac{\partial \mathcal{H}}{\partial \mu_\gamma},$$

is the Poisson bracket.

For the particular case of the cotangent bundle see [81, 90, 47]

1.5.5 Duality between the Lagrangian and Hamiltonian formalism

We have a local diffeomorphism Φ from E^* to E , locally given by

$$x^i = q^i, \quad y^\alpha = \xi^\alpha(q, \mu) = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}. \quad (1.5.36)$$

and its inverse Φ^{-1} has the following local coordinates expression

$$q^i = x^i, \quad \mu_\alpha = \zeta_\alpha(x, y) = \frac{\partial \mathcal{L}}{\partial y^\alpha}, \quad (1.5.37)$$

where

$$\mathcal{L}(x, y) = \zeta_\alpha y^\alpha - \mathcal{H}(x, \mu),$$

and the components $\zeta_\alpha(x, y)$ define an 1-section on E .

From the condition for Φ^{-1} to be the inverse of Φ we get the following formulas

$$\begin{aligned} \mathcal{V}_\beta(\zeta_\alpha) \circ \Phi &= g_{\alpha\beta}, \\ \mathcal{X}_\beta(\zeta_\alpha) \circ \Phi &= -g_{\alpha\gamma} \mathcal{Q}_\beta(\xi^\gamma), \\ \Phi_* \mathcal{P}^\alpha &= (g^{\alpha\beta} \circ \Phi^{-1}) \mathcal{V}_\beta, \\ \Phi_*(\mathcal{Q}_\alpha) &= \mathcal{X}_\alpha + (\mathcal{Q}_\alpha(\xi^\beta) \circ \Phi^{-1}) \mathcal{V}_\beta, \\ \Phi_*^{-1}(\mathcal{V}_\alpha) &= g_{\alpha\beta} \mathcal{P}^\beta, \\ \Phi_*^{-1}(\mathcal{X}_\alpha) &= \mathcal{Q}_\alpha - g_{\gamma\epsilon} \mathcal{Q}_\alpha(\xi^\epsilon) \mathcal{P}^\gamma, \end{aligned} \quad (1.5.38)$$

where Φ_* is tangent map of Φ (see [64]) and $g^{\alpha\beta} = \partial \xi^\alpha / \partial \mu_\beta$, $g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha$.

Theorem 1.5.3 *Let ρ be a \mathcal{J} -regular section on $\mathcal{T}E^*$ induced by the hyperregular Hamiltonian \mathcal{H} , and $\Phi : E^* \rightarrow E$ the global diffeomorphism given by (1.5.36), then the section $\Phi_* \rho$ is a semispray on $\mathcal{T}E$ whose induced connection N is the image by Φ of the connection \mathcal{N} defined by ρ .*

Proof. Let us consider $\rho = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ and we obtain the semispray ϑ

$$\vartheta = \Phi_* \rho = \xi^\alpha (\mathcal{X}_\alpha + (\mathcal{Q}_\alpha(\xi^\beta) \circ \Phi^{-1}) \mathcal{V}_\beta) + \rho_\alpha (g^{\alpha\beta} \circ \Phi^{-1}) \mathcal{V}_\beta = y^\alpha \mathcal{X}_\alpha + \vartheta^\alpha \mathcal{V}_\alpha,$$

where ϑ^α is given by

$$\vartheta^\alpha \circ \Phi = \xi^\beta \mathcal{Q}_\beta(\xi^\alpha) + \rho_\beta \mathcal{P}^\beta(\xi^\alpha) = \rho(\xi^\alpha).$$

Considering $\tilde{\Phi}$ the map induced by Φ at the level of tensors and using (1.5.38) we obtain

$$\tilde{\Phi} \mathcal{J} = J,$$

which yields

$$N = -\mathcal{L}_{\vartheta}J = -\mathcal{L}_{\Phi_*\rho}\tilde{\Phi}\mathcal{J} = -\tilde{\Phi}(\mathcal{L}_{\rho}\mathcal{J}) = \tilde{\Phi}\mathcal{N},$$

thus N is the image of \mathcal{N} by $\tilde{\Phi}$. \square

This theorem shows that the splitting $V\mathcal{T}E^* \oplus H\mathcal{T}E^*$ defined by \mathcal{N} is mapped by Φ_* into the splitting $V\mathcal{T}E \oplus H\mathcal{T}E$ defined by N , and it results:

Corollary 1.5.2 *The following equalities hold*

$$\Phi_*\delta_\alpha^* = \delta_\alpha, \quad \Phi_*^{-1}\delta_\alpha = \delta_\alpha^*,$$

$$\mathcal{N}_{\alpha\beta}(q, \mu) = -(N_\alpha^\gamma(x, y) + \mathcal{Q}_\alpha(\xi^\gamma))g_{\beta\gamma}.$$

$$\mathcal{R}_{\alpha\beta\gamma}g^{\alpha\varepsilon} = \mathcal{R}_{\beta\gamma}^\varepsilon \circ \Phi, \quad \mathcal{R}_{\beta\gamma}^\varepsilon \frac{\partial \zeta_\alpha}{\partial y^\varepsilon} = \mathcal{R}_{\alpha\beta\gamma} \circ \Phi^{-1},$$

$$N_\alpha^\beta \circ \Phi = -\delta_\alpha^*(\xi^\beta), \quad \mathcal{N}_{\alpha\beta} \circ \Phi^{-1} = -\delta_\alpha(\zeta_\beta),$$

Proof. We have on one hand

$$\Phi_*^{-1}(\delta_\alpha) = \delta_\alpha^* = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\gamma}\mathcal{P}^\gamma,$$

and on the other hand

$$\Phi_*^{-1}(\delta_\alpha) = \Phi_*^{-1}(\mathcal{X}_\alpha - N_\alpha^\beta \mathcal{V}_\beta) = \mathcal{Q}_\alpha - g_{\gamma\varepsilon} \mathcal{Q}_\alpha(\xi^\varepsilon) \mathcal{P}^\gamma - N_\alpha^\beta g_{\beta\gamma} \mathcal{P}^\gamma.$$

Therefore, we get $\mathcal{N}_{\alpha\gamma} = -(N_\alpha^\varepsilon + \mathcal{Q}_\alpha(\xi^\varepsilon))g_{\gamma\varepsilon}$. Next, since Φ^{-1} is a diffeomorphism, we obtain

$$[\Phi_*^{-1}(\delta_\alpha), \Phi_*^{-1}(\delta_\beta)]_{\mathcal{T}E^*} = L_{\alpha\beta}^\gamma \Phi_*^{-1}(\delta_\gamma) + \mathcal{R}_{\alpha\beta}^\gamma \Phi_*^{-1}(\mathcal{V}_\gamma),$$

$$[\Phi_*^{-1}(\delta_\alpha), \Phi_*^{-1}(\delta_\beta)]_{\mathcal{T}E^*} = [\delta_\alpha^*, \delta_\beta^*]_{\mathcal{T}E^*} = L_{\alpha\beta}^\gamma \delta_\gamma^* + \mathcal{R}_{\alpha\beta\gamma} \mathcal{P}^\gamma,$$

and using (1.5.38) it results the relation between the curvature tensors of connections N on $\mathcal{T}E$ and \mathcal{N} on $\mathcal{T}E^*$. By direct computations the other relations are obtained. \square

Let us consider \mathcal{S} a semispray on $\mathcal{T}E$ and $\Phi^{-1} : E \rightarrow E^*$ the diffeomorphism given by (1.5.37). Then we set [109]:

Theorem 1.5.4 *The section $\rho = \Phi_*^{-1}\mathcal{S}$ is a semi-Hamiltonian section on $\mathcal{T}E^*$ if and only if the nonlinear connection on $\mathcal{T}E$ induced by semispray is the canonical nonlinear connection induced by the regular Lagrangian.*

Proof. Considering $\mathcal{S} = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha \mathcal{V}_\alpha$ we have from (1.5.38)

$$\Phi_*^{-1}\mathcal{S} = \xi^\alpha \mathcal{Q}_\alpha + (-\xi^\alpha g_{\gamma\varepsilon} \mathcal{Q}_\alpha(\xi^\varepsilon) + \mathcal{S}^\alpha g_{\alpha\gamma}) \mathcal{P}^\gamma.$$

The conditions (1.5.28) *i*), *ii*) and (1.5.38) lead to

$$g^{\theta\beta} (\mathcal{X}_\theta(\zeta_\alpha) - \mathcal{X}_\alpha(\zeta_\theta) + \mathcal{V}_\theta(\mathcal{S}^\varepsilon)g_{\varepsilon\alpha} + \xi^\varepsilon \mathcal{X}_\varepsilon(g_{\alpha\theta}) + \mathcal{S}^\varepsilon \mathcal{V}_\varepsilon(g_{\alpha\theta}) + \zeta_\varepsilon L_{\alpha\theta}^\varepsilon) + y^\varepsilon L_{\alpha\varepsilon}^\beta = 0,$$

and using (1.3.25) it follows

$$\mathcal{X}_\alpha(\zeta_\theta) - \mathcal{X}_\theta(\zeta_\alpha) = \mathcal{S}(g_{\alpha\theta}) - 2N_\theta^\varepsilon g_{\varepsilon\alpha} + y^\beta L_{\theta\beta}^\varepsilon g_{\varepsilon\alpha} + y^\beta L_{\alpha\beta}^\varepsilon g_{\varepsilon\theta} + \zeta_\varepsilon L_{\alpha\theta}^\varepsilon,$$

which concludes that (change α with θ and totalizing)

$$\mathcal{S}(g_{\alpha\theta}) - N_\theta^\varepsilon g_{\varepsilon\alpha} - N_\alpha^\varepsilon g_{\varepsilon\theta} + y^\beta (L_{\theta\beta}^\varepsilon g_{\varepsilon\alpha} + L_{\alpha\beta}^\varepsilon g_{\varepsilon\theta}) = 0 \stackrel{(1.4.19)}{\Leftrightarrow} \nabla g(\mathcal{V}_\alpha, \mathcal{V}_\theta) = 0,$$

and

$$\mathcal{X}_\alpha(\zeta_\theta) - \mathcal{X}_\theta(\zeta_\alpha) = N_\theta^\varepsilon g_{\varepsilon\alpha} - N_\alpha^\varepsilon g_{\varepsilon\theta} + \zeta_\varepsilon L_{\alpha\theta}^\varepsilon,$$

which is equivalent with

$$N_\theta^\varepsilon g_{\varepsilon\alpha} - N_\alpha^\varepsilon g_{\varepsilon\theta} + \frac{\partial^2 L}{\partial x^i \partial y^\alpha} \sigma_\beta^i - \frac{\partial^2 L}{\partial x^i \partial y^\beta} \sigma_\alpha^i - \frac{\partial L}{\partial y^\varepsilon} L_{\alpha\beta}^\varepsilon = 0 \stackrel{(1.4.20)}{\Leftrightarrow} \omega_L(h\rho, h\nu) = 0.$$

From the Theorem 1.4.2 we get that the section $\rho = \Phi_*^{-1}\mathcal{S}$ is semi-Hamiltonian if and only if the connection induced by semispray is the canonical nonlinear connection (1.4.18). When $\rho = \Phi_*^{-1}\mathcal{S}$ is a Hamiltonian section we have the same result, but moreover, the condition (1.5.28) *iii*) seems to lead to the third Helmholtz condition on a Lie algebroid (1.4.22) and the work is in progress. \square

1.6 Dynamical covariant derivative and metric non-linear connection on $\mathcal{T}E^*$

Definition 1.6.1 A map $\nabla : \mathfrak{T}(\mathcal{T}E^* \setminus \{0\}) \rightarrow \mathfrak{T}(\mathcal{T}E^* \setminus \{0\})$ is said to be a tensor derivation on $\mathcal{T}E^* \setminus \{0\}$ if the following conditions are satisfied:

- i) ∇ is \mathbb{R} -linear
- ii) ∇ is type preserving, i.e. $\nabla \mathfrak{T}_s^r(\mathcal{T}E^* \setminus \{0\}) \subset \mathfrak{T}_s^r(\mathcal{T}E^* \setminus \{0\})$, for each $(r, s) \in \mathbb{N} \times \mathbb{N}$
- iii) ∇ obeys the Leibnitz rule $\nabla(P \otimes S) = \nabla P \otimes S + P \otimes \nabla S$ for any tensors P, S on $\mathcal{T}E^* \setminus \{0\}$
- iv) ∇ commutes with any contractions.

We consider the \mathbb{R} -linear map $\nabla_\rho : \Gamma(\mathcal{T}E^* \setminus \{0\}) \rightarrow \Gamma(\mathcal{T}E^* \setminus \{0\})$ by

$$\nabla_\rho X = h[\rho, hX]_{\mathcal{T}E^*} + v[\rho, vX]_{\mathcal{T}E^*}, \quad \forall X \in \Gamma(\mathcal{T}E^* \setminus \{0\}). \quad (1.6.1)$$

where ρ is a \mathcal{J} -regular section and it follows that

$$\nabla_\rho(fX) = \rho(f)X + f\nabla_\rho X, \quad \forall f \in C^\infty(\mathcal{T}E^* \setminus \{0\}), \quad (1.6.2)$$

Any tensor derivation on $\mathcal{T}E^* \setminus \{0\}$ is completely determined by its actions on smooth functions and sections on $\mathcal{T}E^* \setminus \{0\}$ (see [117] generalized Willmore's theorem, p. 1217). Therefore, there exists a unique tensor derivation ∇_ρ on $\mathcal{T}E^* \setminus \{0\}$ such that

$$\nabla_\rho|_{C^\infty(\mathcal{T}E^* \setminus \{0\})} = \rho, \quad \nabla_\rho|_{\Gamma(\mathcal{T}E^* \setminus \{0\})} = \nabla_0.$$

We will call the tensor derivation ∇_ρ , the *dynamical covariant derivative* induced by the \mathcal{J} -regular section ρ and a nonlinear connection \mathcal{N} .

Proposition 1.6.1 The following equations hold

$$[\rho, \mathcal{P}^\beta]_{\mathcal{T}E^*} = -t^{\alpha\beta}\delta_\alpha^* + \left(t^{\alpha\beta}\mathcal{N}_{\alpha\gamma} - \frac{\partial\rho_\gamma}{\partial\mu_\beta}\right)\mathcal{P}^\gamma, \quad (1.6.3)$$

$$[\rho, \delta_\beta^*]_{\mathcal{T}E^*} = -(\delta_\beta^*(\xi^\alpha) + \xi^\varepsilon L_{\varepsilon\beta}^\alpha)\delta_\alpha^* + \mathcal{R}_{\beta\gamma}\mathcal{P}^\gamma, \quad (1.6.4)$$

where

$$\mathcal{R}_{\beta\gamma} = \delta_\beta^*(\xi^\alpha)\mathcal{N}_{\alpha\gamma} + \rho(\mathcal{N}_{\beta\gamma}) - \delta_\beta^*(\rho_\gamma) - \xi^\varepsilon L_{\varepsilon\beta}^\alpha\mathcal{N}_{\alpha\gamma}, \quad (1.6.5)$$

The action of ∇_ρ on the Berwald basis has the form

$$\nabla_\rho \mathcal{P}^\beta = v[\rho, \mathcal{P}^\beta]_{\mathcal{T}E^*} = \left(t^{\alpha\beta}\mathcal{N}_{\alpha\gamma} - \frac{\partial\rho_\gamma}{\partial\mu_\beta}\right)\mathcal{P}^\gamma,$$

$$\nabla_\rho \delta_\beta^* = h[\rho, \delta_\beta^*]_{\mathcal{T}E^*} = -(\delta_\beta^*(\xi^\alpha) + \xi^\varepsilon L_{\varepsilon\beta}^\alpha)\delta_\alpha^*.$$

For a pseudo-Riemannian metric g on $\mathcal{T}E^*$ the action of ∇_ρ is given by

$$\nabla_\rho g(X, Y) = \rho(g(X, Y)) - g(\nabla_\rho X, Y) - g(X, \nabla_\rho Y), \quad (1.6.6)$$

which in local coordinates leads to

$$g_{/'}^{\alpha\beta} := \nabla_\rho g(\mathcal{P}^\alpha, \mathcal{P}^\beta) = \rho(g^{\alpha\beta}) - g^{\varepsilon\beta} \left(t^{\alpha\gamma} \mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\alpha} \right) - g^{\varepsilon\alpha} \left(t^{\beta\gamma} \mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\beta} \right).$$

Definition 1.6.2 A nonlinear connection is called metric or compatible with the metric tensor g if $\nabla_\rho g = 0$, for all \mathcal{J} -regular sections ρ , that is

$$\rho(g(X, Y)) = g(\nabla_\rho X, Y) + g(X, \nabla_\rho Y), \quad \forall X, Y \in \Gamma(V\mathcal{T}E^*). \quad (1.6.7)$$

Theorem 1.6.1 The connection $\tilde{\mathcal{N}}$ with the coefficients

$$\widetilde{\mathcal{N}}_{\alpha\beta} = \mathcal{N}_{\alpha\beta} + \frac{1}{2} t_{\alpha\varepsilon} g_{\beta\gamma} g_{/'}^{\varepsilon\gamma}, \quad (1.6.8)$$

is a metric nonlinear connection.

Proof. Let us consider the dynamical covariant derivative induced by ρ and $\tilde{\mathcal{N}}$ given by

$$\nabla_\rho g(\mathcal{P}^\alpha, \mathcal{P}^\beta) = \rho(g^{\alpha\beta}) - g^{\varepsilon\beta} \left(t^{\alpha\gamma} \tilde{\mathcal{N}}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\alpha} \right) - g^{\varepsilon\alpha} \left(t^{\beta\gamma} \tilde{\mathcal{N}}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\beta} \right),$$

and using (1.6.8) it follows

$$\nabla_\rho g(\mathcal{P}^\alpha, \mathcal{P}^\beta) = g_{/'}^{\alpha\beta} - \frac{1}{2} g^{\varepsilon\beta} t^{\alpha\gamma} t_{\gamma r} g_{\varepsilon p} g_{/'}^{rp} - \frac{1}{2} g^{\alpha\varepsilon} t^{\beta\gamma} t_{\gamma r} g_{\varepsilon p} g_{/'}^{rp} = 0,$$

that is $\tilde{\mathcal{N}}$ is a metric nonlinear connection. \square

1.6.1 Nonlinear connection induced by a \mathcal{J} -regular section

If \mathcal{J} is an adapted tangent structure and ρ is a \mathcal{J} -regular section then [50]

$$\mathcal{N} = -\mathcal{L}_\rho \mathcal{J},$$

is a nonlinear connection on $\mathcal{T}E^*$ with local coefficients given by (1.5.26)

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2} \left(t_{\alpha\gamma} \frac{\partial \rho_\beta}{\partial \mu_\gamma} - \sigma_\alpha^i t_{\gamma\beta} \frac{\partial \xi^\gamma}{\partial q^i} - \rho(t_{\alpha\beta}) + \xi^\gamma t_{\varepsilon\beta} L_{\gamma\alpha}^\varepsilon \right).$$

Definition 1.6.3 *The Jacobi endomorphism ψ is given by*

$$\psi = v[\rho, hX]_{\mathcal{T}E^*}.$$

Locally, from (1.6.4) we obtain that $\psi = \mathcal{R}_{\alpha\beta} \mathcal{Q}^\alpha \otimes \mathcal{P}^\beta$, where $\mathcal{R}_{\alpha\beta}$ is given by (1.6.8). $\mathcal{R}_{\alpha\beta}$ are the local coefficients of the Jacobi endomorphism.

Proposition 1.6.2 *The following result holds*

$$\psi = i_\rho \Omega + v[v\rho, hX]_{\mathcal{T}E^*}.$$

Proof. Indeed, $\psi(X) = v[\rho, hX]_{\mathcal{T}E^*} = v[h\rho, hX]_{\mathcal{T}E^*} + v[v\rho, hX]_{\mathcal{T}E^*}$ and $\Omega(\rho, X) = v[h\rho, hX]_{\mathcal{T}E^*}$, that is $\psi(X) = \Omega(\rho, X) + v[v\rho, hX]_{\mathcal{T}E^*}$. \square

Remark 1.6.1 *If ρ is a horizontal section $\rho = h\rho$, then we obtain*

$$\psi = i_\rho \Omega,$$

and locally it follows

$$\rho_\gamma = \xi^\alpha N_{\alpha\gamma},$$

which yields

$$\mathcal{R}_{\alpha\beta} = \mathcal{R}_{\varepsilon\alpha\beta} \xi^\varepsilon. \quad (1.6.9)$$

and the local relation between the Jacobi endomorphism and the curvature tensor of the nonlinear connection is obtained.

1.6.2 Hamiltonian case

In what follows, we consider a regular Hamiltonian $\mathcal{H} : E^* \rightarrow \mathbb{R}$, that is the matrix

$$g^{\alpha\beta}(q, \mu) = \frac{\partial^2 \mathcal{H}}{\partial \mu_\alpha \partial \mu_\beta},$$

is nondegenerate. This regular Hamiltonian determines a symmetric canonical nonlinear connection given by (1.5.35). We have the following result:

Theorem 1.6.2 *The canonical nonlinear connection induced by a regular Hamiltonian \mathcal{H} is a metric nonlinear connection.*

Proof. Introducing the coefficients (1.5.35) into the expression of the dynamical covariant derivative and using (1.5.33) we obtain

$$\begin{aligned} \nabla_\rho g(\mathcal{P}^\alpha, \mathcal{P}^\beta) &= \sigma_\varepsilon^i \frac{\partial g^{\alpha\beta}}{\partial q^i} \frac{\partial \mathcal{H}}{\partial \mu_\varepsilon} - \sigma_\varepsilon^i \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial g^{\alpha\beta}}{\partial \mu_\varepsilon} - \mu_\gamma L_{\varepsilon\delta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\delta} \frac{\partial g^{\alpha\beta}}{\partial \mu_\varepsilon} \\ &- \frac{1}{2} \left(g^{\varepsilon\beta} g^{\alpha\gamma} + g^{\varepsilon\alpha} g^{\beta\gamma} \right) \left[\sigma_l^i \left(\frac{\partial g_{\gamma\varepsilon}}{\partial \mu_l} \frac{\partial \mathcal{H}}{\partial q^i} - \frac{\partial g_{\gamma\varepsilon}}{\partial q^i} \frac{\partial \mathcal{H}}{\partial \mu_l} \right) - \sigma_\varepsilon^i \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_l} g_{\gamma l} \right. \\ &- \sigma_\gamma^i \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_l} g_{\varepsilon l} + \mu_l L_{rs}^l \frac{\partial \mathcal{H}}{\partial \mu_r} \frac{\partial g_{\gamma\varepsilon}}{\partial \mu_s} + \mu_l L_{\gamma\varepsilon}^l + \frac{\partial \mathcal{H}}{\partial \mu_\delta} \left(g_{\gamma l} L_{\delta\varepsilon}^l + g_{\varepsilon l} L_{\delta\gamma}^l \right) \Big] \\ &- g^{\varepsilon\beta} \sigma_\varepsilon^i \frac{\partial^2 \mathcal{H}}{\partial \mu_\alpha \partial q^i} - g^{\varepsilon\beta} L_{\varepsilon\delta}^\alpha \frac{\partial \mathcal{H}}{\partial \mu_\delta} - g^{\varepsilon\beta} g^{\alpha\delta} \mu_\gamma L_{\varepsilon\delta}^\gamma \\ &- g^{\varepsilon\alpha} \sigma_\varepsilon^i \frac{\partial^2 \mathcal{H}}{\partial \mu_\beta \partial q^i} - g^{\varepsilon\alpha} L_{\varepsilon\delta}^\beta \frac{\partial \mathcal{H}}{\partial \mu_\delta} - g^{\varepsilon\alpha} g^{\beta\delta} \mu_\gamma L_{\varepsilon\delta}^\gamma. \end{aligned}$$

From the equalities

$$g^{\varepsilon\beta} g^{\alpha\gamma} \frac{\partial g_{\gamma\varepsilon}}{\partial \mu_l} = -g^{\varepsilon\beta} g_{\gamma\varepsilon} \frac{\partial g^{\alpha\gamma}}{\partial \mu_l} = -\frac{\partial g^{\alpha\beta}}{\partial \mu_l}, \quad (1.6.10)$$

$$g^{\varepsilon\beta} g^{\alpha\gamma} \frac{\partial g_{\gamma\varepsilon}}{\partial q^i} = -g^{\varepsilon\beta} g_{\gamma\varepsilon} \frac{\partial g^{\alpha\gamma}}{\partial q^i} = -\frac{\partial g^{\alpha\beta}}{\partial q^i}, \quad L_{\varepsilon\delta}^\gamma = -L_{\delta\varepsilon}^\gamma \quad (1.6.11)$$

by direct computation, it follows that $\nabla_\rho g(\mathcal{P}^\alpha, \mathcal{P}^\beta) = 0$, which ends the proof. \square

Theorem 1.6.3 *The canonical nonlinear connection induced by a regular Hamiltonian is the unique metric and symmetric nonlinear connection.*

Proof. Let us consider a metric and symmetric nonlinear connection \mathcal{N} with the coefficients $\mathcal{N}_{\gamma\varepsilon}$. Then we have

$$\rho_{\mathcal{H}}(g^{\alpha\beta}) = g^{\varepsilon\beta} \left(g^{\alpha\gamma} \mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_{\varepsilon}}{\partial \mu_{\alpha}} \right) + g^{\varepsilon\alpha} \left(g^{\beta\gamma} \mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_{\varepsilon}}{\partial \mu_{\beta}} \right)$$

and using (1.5.33) we obtain

$$\begin{aligned} & \frac{\partial \mathcal{H}}{\partial \mu_l} \sigma_l^i \frac{\partial g^{\alpha\beta}}{\partial q^i} - \left(\sigma_l^i \frac{\partial \mathcal{H}}{\partial q^i} + \mu_{\gamma} L_{l\varepsilon}^{\gamma} \frac{\partial \mathcal{H}}{\partial \mu_{\varepsilon}} \right) \frac{\partial g^{\alpha\beta}}{\partial \mu_l} = \\ & g^{\varepsilon\beta} g^{\alpha\gamma} \mathcal{N}_{\gamma\varepsilon} + g^{\varepsilon\alpha} g^{\beta\gamma} \mathcal{N}_{\gamma\varepsilon} + g^{\gamma\beta} \sigma_{\gamma}^i \frac{\partial^2 \mathcal{H}}{\partial \mu_{\alpha} \partial q^i} + g^{\gamma\beta} L_{\gamma\tau}^{\alpha} \frac{\partial \mathcal{H}}{\partial \mu_{\tau}} + \\ & g^{\gamma\beta} \mu_l L_{\gamma\tau}^l \frac{\partial^2 \mathcal{H}}{\partial \mu_{\alpha} \partial \mu_{\tau}} + g^{\varepsilon\alpha} \sigma_{\varepsilon}^i \frac{\partial^2 \mathcal{H}}{\partial \mu_{\beta} \partial q^i} + g^{\varepsilon\alpha} L_{\varepsilon\tau}^{\beta} \frac{\partial \mathcal{H}}{\partial \mu_{\tau}} + g^{\varepsilon\alpha} \mu_l L_{\varepsilon\tau}^l \frac{\partial^2 \mathcal{H}}{\partial \mu_{\beta} \partial \mu_{\tau}}. \end{aligned}$$

But the connection is symmetric (1.5.10) and using (1.6.10), (1.6.11) we get

$$\begin{aligned} & \sigma_l^i \frac{\partial \mathcal{H}}{\partial q^i} \frac{\partial g_{\varepsilon\gamma}}{\partial \mu_l} - \sigma_l^i \frac{\partial \mathcal{H}}{\partial \mu_l} \frac{\partial g_{\varepsilon\gamma}}{\partial q^i} + \mu_{\tau} L_{l\alpha}^{\tau} \frac{\partial \mathcal{H}}{\partial \mu_{\alpha}} \frac{\partial g_{\varepsilon\gamma}}{\partial \mu_l} = \\ & 2\mathcal{N}_{\varepsilon\gamma} + \mu_{\tau} L_{\gamma\varepsilon}^{\tau} - g_{\alpha\varepsilon} \sigma_{\gamma}^i \frac{\partial^2 \mathcal{H}}{\partial \mu_{\alpha} \partial q^i} - g_{\alpha\gamma} \sigma_{\varepsilon}^i \frac{\partial^2 \mathcal{H}}{\partial \mu_{\alpha} \partial q^i} - g_{\alpha\varepsilon} L_{\gamma l}^{\alpha} \frac{\partial \mathcal{H}}{\partial \mu_l} - g_{\alpha\gamma} L_{\varepsilon l}^{\alpha} \frac{\partial \mathcal{H}}{\partial \mu_l}, \end{aligned}$$

and we get the coefficients (1.5.35), which ends the proof. \square

For the particular case of the cotangent bundle see [100, 105]

1.7 Poisson-Lie algebroids

Poisson manifolds were introduced by A. Lichnerowicz in his famous paper [66] and their properties were later investigated by A. Weinstein [124]. The Poisson manifolds are the smooth manifolds equipped with a Poisson bracket on their ring of functions. We remark that the cotangent bundle of a Poisson manifold has a natural structure of Lie algebroid. In this chapter we study the Poisson structures on Lie algebroids [97, 99, 107].

1.7.1 Poisson structures on Lie algebroids

The Schouten-Nijenhuis bracket is given by [120]

$$[X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q] = (-1)^{p+1} \sum_{i=1}^p \sum_{j=1}^q (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \wedge \hat{X}_i \wedge \dots \wedge X_p \wedge Y_1 \wedge \dots \wedge \hat{Y}_j \wedge \dots \wedge Y_q$$

Let us consider the bivector (i.e. contravariant, skew-symmetric, 2-section) $\Pi \in \Gamma(\wedge^2 E)$ given by the expression

$$\Pi = \frac{1}{2} \pi^{\alpha\beta}(x) s_\alpha \wedge s_\beta. \quad (1.7.1)$$

Definition 1.7.1 *The bivector Π is a Poisson bivector on E if and only if $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is Schouten-Nijenhuis bracket.*

Proposition 1.7.1 *Locally the condition $[\Pi, \Pi] = 0$ is expressed as*

$$\sum_{(\alpha, \varepsilon, \delta)} (\pi^{\alpha\beta} \sigma_\beta^i \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} + \pi^{\alpha\beta} \pi^{\gamma\delta} L_{\beta\gamma}^\varepsilon) = 0 \quad (1.7.2)$$

If Π is a Poisson bivector then the pair (E, Π) is called the *Poisson-Lie algebroid*. The Poisson bracket on M is given by

$$\{f_1, f_2\} = \Pi(d^E f_1, d^E f_2), \quad f_1, f_2 \in C^\infty(M)$$

We also have the bundle map $\pi^\# : E^* \rightarrow E$ defined by

$$\pi^\# \rho = i_\rho \Pi, \quad \rho \in \Gamma(E^*).$$

Let us consider the bracket

$$[\rho, \theta]_\pi = \mathcal{L}_{\pi^\# \rho} \theta - \mathcal{L}_{\pi^\# \theta} \rho - d^E(\Pi(\rho, \theta)),$$

where \mathcal{L} is Lie derivative and $\rho, \theta \in \Gamma(E^*)$. With respect to this bracket and the usual Lie bracket on vector fields, the map $\tilde{\sigma} : E^* \rightarrow TM$ given by

$$\tilde{\sigma} = \sigma \circ \pi^\#,$$

is a Lie algebra homomorphism

$$\tilde{\sigma}[\rho, \theta]_\pi = [\tilde{\sigma}\rho, \tilde{\sigma}\theta].$$

The bracket $[\cdot, \cdot]_\pi$ satisfies also the Leibniz rule

$$[\rho, f\theta]_\pi = f[\rho, \theta]_\pi + \tilde{\sigma}(\rho)(f)\theta,$$

and it results that $(E^*, [\cdot, \cdot]_\pi, \tilde{\sigma})$ is a Lie algebroid [61].

Next, we can define the contravariant exterior differential $d^\pi : \bigwedge^k(E^*) \rightarrow \bigwedge^{k+1}(E^*)$ by

$$\begin{aligned} d^\pi \omega(s_1, \dots, s_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \tilde{\sigma}(s_i) \omega(s_1, \dots, \hat{s}_i, \dots, s_{k+1}) + \\ &\quad + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega([s_i, s_j]_\pi, s_1, \dots, \hat{s}_i, \dots, \hat{s}_j, \dots, s_{k+1}). \end{aligned}$$

Accordingly, we get the cohomology of Lie algebroid E^* with the anchor $\tilde{\sigma}$ and the bracket $[\cdot, \cdot]_\pi$ which generalize the Poisson cohomology of Lichnerowicz for Poisson manifolds [120].

In the following we deal with the notion of contravariant connection on Lie algebroids, which generalize the similar notion on Poisson manifolds [39], [66].

Definition 1.7.2 *If $\rho, \theta \in \Gamma(E^*)$ and $\Phi, \Psi \in \Gamma(E)$ then the linear contravariant connection is an application $D : \Gamma(E^*) \times \Gamma(E) \rightarrow \Gamma(E)$ which satisfies the relations*

- i) $D_{\rho+\theta}\Phi = D_\rho\Phi + D_\theta\Phi,$
- ii) $D_\rho(\Phi + \Psi) = D_\rho\Phi + D_\rho\Psi,$
- iii) $D_{f\rho}\Phi = fD_\rho\Phi,$
- iv) $D_\rho(f\Phi) = fD_\rho\Phi + \tilde{\sigma}(\rho)(f)\Phi, \quad f \in C^\infty(M).$

The contravariant connection induces a contravariant derivative $D_\alpha : \Gamma(E) \rightarrow \Gamma(E)$ such that the following equalities are fulfilled

$$D_{f_1\alpha_1+f_2\alpha_2} = f_1D_{\alpha_1} + f_2D_{\alpha_2}, \quad f_i \in C^\infty(M), \quad \alpha_i \in \Gamma(E^*)$$

$$D_\rho(f\theta) = fD_\rho\theta + \tilde{\sigma}(\rho)(f)\theta, \quad f \in C^\infty(M), \quad \rho, \theta \in \Gamma(E^*)$$

In the case where the contravariant connection D is induced by a covariant connection ∇ on a Lie algebroid E (see [30]) we have $D_\rho = \nabla_{\pi^\#\rho}$.

Definition 1.7.3 *The torsion and curvature of linear contravariant connection are given by*

$$T(\rho, \theta) = D_\rho \theta - D_\theta \rho - [\rho, \theta]_\pi,$$

$$R(\rho, \theta)\mu = D_\rho D_\theta \mu - D_\theta D_\rho \mu - D_{[\rho, \theta]_\pi} \mu,$$

where $\rho, \theta, \mu \in \Gamma(E^*)$.

The curvature tensor satisfies the equalities

$$R(\rho, \theta) = -R(\theta, \rho), \quad R(f\rho, \theta) = fR(\rho, \theta).$$

The Bianchi identities have the following form

$$\sum_{\rho, \theta, \mu} (D_\rho R(\theta, \mu) + R(T(\rho, \theta), \mu)) = 0,$$

$$\sum_{\rho, \theta, \mu} (R(\rho, \theta)\mu - T(T(\rho, \theta), \mu) - D_\rho T(\theta, \mu)) = 0.$$

In local coordinates we define the Christoffel symbols $\Gamma_\gamma^{\alpha\beta}$ by the formula

$$D_{s^\alpha} s^\beta = \Gamma_\gamma^{\alpha\beta} s^\gamma,$$

and under a change of coordinates

$$\begin{cases} x^{i'} = x^{i'}(x^i), & i, i' = \overline{1, n} \text{ on } M \\ y^{\alpha'} = A_{\alpha}^{\alpha'} y^\alpha, & \alpha, \alpha' = \overline{1, m} \text{ on } E, \end{cases} \quad (1.7.3)$$

corresponding to a new base $s^{\alpha'} = A_{\alpha}^{\alpha'} s^\alpha$, these symbols transform according to

$$\Gamma_{\gamma'}^{\alpha'\beta'} = A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} A_{\gamma}^{\gamma'} \Gamma_{\gamma}^{\alpha\beta} + A_{\alpha}^{\alpha'} A_{\gamma'}^{\gamma} \sigma_{\varepsilon}^i \frac{\partial A_{\gamma}^{\beta'}}{\partial x^i} \pi^{\alpha\varepsilon}. \quad (1.7.4)$$

If we denote $T(s^\alpha, s^\beta) = T_\gamma^{\alpha\beta} s^\gamma$ and $R(s^\alpha, s^\beta)s^\gamma = R_\delta^{\alpha\beta\gamma} s^\delta$ then, under a change of coordinates, we obtain that

$$T_{\gamma'}^{\alpha'\beta'} = A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} A_{\gamma}^{\gamma'} T_{\gamma}^{\alpha\beta}, \quad R_{\delta'}^{\alpha'\beta'\gamma'} = A_{\alpha}^{\alpha'} A_{\beta}^{\beta'} A_{\gamma}^{\gamma'} A_{\delta}^{\delta'} R_{\delta}^{\alpha\beta\gamma}.$$

Proposition 1.7.2 *The local components of torsion and curvature of linear contravariant connection are*

$$T_{\varepsilon}^{\alpha\beta} = \Gamma_{\varepsilon}^{\alpha\beta} - \Gamma_{\varepsilon}^{\beta\alpha} - \pi^{\alpha\gamma} L_{\gamma\varepsilon}^{\beta} + \pi^{\beta\gamma} L_{\gamma\varepsilon}^{\alpha} - \sigma_{\varepsilon}^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i},$$

$$R_{\delta}^{\alpha\beta\gamma} = \Gamma_{\delta}^{\alpha\varepsilon} \Gamma_{\varepsilon}^{\beta\gamma} - \Gamma_{\delta}^{\beta\varepsilon} \Gamma_{\varepsilon}^{\alpha\gamma} + \pi^{\alpha\varepsilon} \sigma_{\varepsilon}^i \frac{\partial \Gamma_{\delta}^{\beta\gamma}}{\partial x^i} - \pi^{\beta\varepsilon} \sigma_{\varepsilon}^i \frac{\partial \Gamma_{\delta}^{\alpha\gamma}}{\partial x^i} + (\pi^{\beta\nu} L_{\nu\varepsilon}^{\alpha} - \pi^{\alpha\nu} L_{\nu\varepsilon}^{\beta} - \sigma_{\varepsilon}^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i}) \Gamma_{\delta}^{\varepsilon\gamma}.$$

Definition 1.7.4 A tensor field T on E is called parallel with respect to D if and only if $DT = 0$.

Definition 1.7.5 A contravariant connection D is called a Poisson connection if the Poisson bivector Π is parallel with respect to D .

Remark 1.7.1 If the Poisson connection D is induced by a covariant connection ∇ (i.e. $D\Pi = 0$, $D_\rho = \nabla_{\pi^\# \rho}$, $\pi^\# D_\rho \phi = \nabla_{\pi^\# \rho} \pi^\# \phi$) then the torsion and curvature tensors of the both connections are related by the following equalities

$$\begin{aligned} T^\nabla(\pi^\# \rho, \pi^\# \theta) &= \pi^\# T^D(\rho, \theta), \\ R^\nabla(\pi^\# \rho, \pi^\# \theta) \pi^\# \mu &= \pi^\# R^D(\rho, \theta) \mu, \quad \forall \rho, \theta, \mu \in \Gamma(E^*). \end{aligned}$$

Let T be a tensor of type (r, s) with the components $T_{j_1 \dots j_s}^{i_1 \dots i_r}$ and $\theta = \theta_\alpha s^\alpha$ a section of E^* . The local coordinates expression of contravariant derivative is given by

$$D_\theta T = \theta_\alpha T_{j_1 \dots j_s}^{i_1 \dots i_r} /^\alpha s_{i_1} \otimes \dots \otimes s_{i_r} \otimes s^{j_1} \otimes \dots \otimes s^{j_s},$$

where

$$T_{j_1 \dots j_s}^{i_1 \dots i_r} /^\alpha = \pi^{\alpha\varepsilon} \sigma_\varepsilon^i \frac{\partial T_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial x^i} + \sum_{a=1}^r \left(\Gamma_\varepsilon^{i_a \alpha} T_{j_1 \dots j_s}^{i_1 \dots \varepsilon \dots i_r} \right) - \sum_{b=1}^s \left(\Gamma_{j_b}^{\varepsilon \alpha} T_{j_1 \dots \varepsilon \dots j_s}^{i_1 \dots i_r} \right),$$

and $/$ denote the contravariant derivative operator.

Let us consider a contravariant connection D with the coefficients $\Gamma_\gamma^{\alpha\beta}$. We have

Proposition 1.7.3 The contravariant connection \overline{D} with the coefficients given by

$$\overline{\Gamma}_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta} - \frac{1}{2} \pi_{\gamma\varepsilon} \pi^{\alpha\varepsilon} /^\beta, \quad (1.7.5)$$

is a Poisson connection.

Proof. Considering $\overline{\Gamma}$ the contravariant derivative operator with respect to contravariant connection \overline{D} , we get

$$\begin{aligned} \pi^{\beta\gamma} \overline{\Gamma}^\alpha &= \pi^{\alpha\varepsilon} \sigma_\varepsilon^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i} + \overline{\Gamma}_\varepsilon^{\beta\alpha} \pi^{\varepsilon\gamma} + \overline{\Gamma}_\varepsilon^{\gamma\alpha} \pi^{\beta\varepsilon} = \\ &= \pi^{\alpha\varepsilon} \sigma_\varepsilon^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i} + \left(\Gamma_\varepsilon^{\beta\alpha} - \frac{1}{2} \pi_{\varepsilon\tau} \pi^{\beta\tau} /^\alpha \right) \pi^{\varepsilon\gamma} + \left(\Gamma_\varepsilon^{\gamma\alpha} - \frac{1}{2} \pi_{\varepsilon\tau} \pi^{\gamma\tau} /^\alpha \right) \pi^{\beta\varepsilon} \\ &= \pi^{\beta\gamma} /^\alpha - \frac{1}{2} \pi^{\beta\gamma} /^\alpha - \frac{1}{2} \pi^{\beta\gamma} /^\alpha = 0. \end{aligned}$$

□

Proposition 1.7.4 a) *The functions*

$$\Gamma_{\gamma}^{\alpha\beta} = \sigma_{\gamma}^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i}, \quad (1.7.6)$$

are the coefficients of a contravariant connection.

b) *The contravariant connection with the coefficients given by (1.7.6) is a Poisson connection if and only if*

$$\sum_{\alpha, \varepsilon, \delta} \pi^{\alpha\beta} \pi^{\gamma\delta} L_{\beta\gamma}^{\varepsilon} = 0. \quad (1.7.7)$$

Proof. a) Using the change of coordinates (1.7.3) and the fact that the structure function σ_{α}^i change by the rule [69]

$$\sigma_{\alpha'}^{i'} A_{\alpha'}^{\alpha'} = \frac{\partial x^{i'}}{\partial x^i} \sigma_{\alpha}^i,$$

we obtain that the coefficients (1.7.6) satisfy the transformation law (1.7.4).

b) We obtain that

$$\pi^{\beta\gamma}/\alpha = \pi^{\alpha\varepsilon} \sigma_{\varepsilon}^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i} - \Gamma_{\varepsilon}^{\beta\alpha} \pi^{\varepsilon\gamma} - \Gamma_{\varepsilon}^{\gamma\alpha} \pi^{\beta\varepsilon},$$

and using the equality (1.7.6), it results

$$\pi^{\beta\gamma}/\alpha = \sum_{(\alpha, \beta, \gamma)} \pi^{\alpha\varepsilon} \sigma_{\varepsilon}^i \frac{\partial \pi^{\beta\gamma}}{\partial x^i}.$$

From the condition $[\Pi, \Pi] = 0$, locally given by the equation (1.7.2), it results that $\pi^{\beta\gamma}/\alpha = 0$ if and only if the required relation is fulfilled. \square

Remark 1.7.2 *Considering*

$$\Gamma_{\gamma}^{\alpha\beta} = \sigma_{\gamma}^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i}$$

in relation (1.7.5) we obtain a Poisson connection \overline{D} with the coefficients

$$\overline{\Gamma}_{\gamma}^{\alpha\beta} = \sigma_{\gamma}^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} - \frac{1}{2} \pi_{\gamma\varepsilon} \pi^{\alpha\varepsilon}/\beta$$

which depends only on the Poisson bivector and structural functions of Lie algebroid.

Proposition 1.7.5 *The set of Poisson connections on Lie algebroid are given by*

$$\overline{\Gamma}_{\gamma}^{\alpha\beta} = \Gamma_{\gamma}^{\alpha\beta} + \Omega_{\gamma\nu}^{\alpha\varepsilon} X_{\varepsilon}^{\nu\beta},$$

where

$$\Omega_{\gamma\nu}^{\alpha\varepsilon} = \frac{1}{2} (\delta_{\nu}^{\alpha} \delta_{\gamma}^{\varepsilon} - \pi_{\gamma\nu} \pi^{\alpha\varepsilon}),$$

and $D(\Gamma_{\gamma}^{\alpha\beta})$ is a Poisson connection with $X_{\varepsilon}^{\delta\beta}$ an arbitrary tensor.

Proof. By straightforward computation it results

$$\begin{aligned}\pi^{\beta\gamma}\overline{\gamma}^\alpha &= \pi^{\alpha\varepsilon}\sigma_\varepsilon^i\frac{\partial\pi^{\beta\gamma}}{\partial x^i} + \overline{\Gamma}_\varepsilon^{\beta\alpha}\pi^{\varepsilon\gamma} + \overline{\Gamma}_\varepsilon^{\gamma\alpha}\pi^{\beta\varepsilon} = \\ \pi^{\beta\gamma}/^\alpha + \frac{1}{2}\pi^{\varepsilon\gamma}(\delta_\nu^\beta\delta_\varepsilon^\theta - \pi_{\varepsilon\nu}\pi^{\beta\theta})X_\theta^{\nu\alpha} + \frac{1}{2}\pi^{\beta\varepsilon}(\delta_\nu^\gamma\delta_\varepsilon^\theta - \pi_{\varepsilon\nu}\pi^{\gamma\theta})X_\theta^{\nu\alpha} = \\ \pi^{\beta\gamma}/^\alpha + \frac{1}{2}\pi^{\theta\gamma}X_\theta^{\beta\alpha} - \frac{1}{2}\pi^{\beta\theta}X_\theta^{\gamma\alpha} + \frac{1}{2}\pi^{\beta\theta}X_\theta^{\gamma\alpha} - \frac{1}{2}\pi^{\theta\gamma}X_\theta^{\beta\alpha} &= 0,\end{aligned}$$

because $\pi^{\beta\gamma}/^\alpha = 0$, which ends the proof. \square

The image of the anchor map $\sigma(E) \subseteq TM$ defines an integrable smooth distribution on M . Therefore, the manifold M is foliated by the integral leaves of $\sigma(E)$, which are called the leaves of the Lie algebroid. A curve $u : [t_0, t_1] \rightarrow E$ is called admissible if $\sigma(u(t)) = \dot{c}(t)$, where $c(t) = \pi(u(t))$ is the base curve on M . It follows that $u(t)$ is admissible if and only if the base curve $c(t)$ lies on a leaf of the Lie algebroid whereas two points can be joint by an admissible curve if and only if they are situated on the same leaf. We can choose a smooth family $t \rightarrow \theta(t) \in E^*$ of 1-form such that $\tilde{\sigma}\theta(t) = \dot{c}(t)$. We shall call the pair $(u(t), \theta(t))$ the *dual curve*.

Definition 1.7.6 Let $(u(t), \theta(t))$ a dual curve on E . We say that $(u(t), \theta(t))$ is a geodesic if

$$(D_\theta\theta)_{u(t)} = 0.$$

In local coordinates we obtain that a curve

$$(u(t), \theta(t)) = (x^1(t), \dots, x^n(t), \theta_1(t), \dots, \theta_m(t))$$

is geodesic if and only if it satisfies the following system of differential equations

$$\begin{cases} \frac{dx^i(t)}{dt} = \sigma_\gamma^i\pi^{\alpha\gamma}(x^1(t), \dots, x^n(t))\theta_\alpha(t) \\ \frac{d\theta_\alpha(t)}{dt} = -\Gamma_\alpha^{\beta\gamma}(x^1(t), \dots, x^n(t))\theta_\beta\theta_\gamma. \end{cases} \quad (1.7.8)$$

1.7.2 Compatible Poisson structures

Let us consider the Poisson bivector on Lie algebroids given by the relation (1.7.1). We obtain

Proposition 1.7.6 *The complete lift of Π on $\mathcal{T}E$ is given by*

$$\Pi^c = \pi^{\alpha\beta} \mathcal{X}_\alpha \wedge \mathcal{V}_\beta + \left(\frac{1}{2} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} - \pi^{\delta\beta} L_{\delta\gamma}^\alpha \right) y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta. \quad (1.7.9)$$

Proof. Using the properties of the vertical and complete lifts we obtain

$$\begin{aligned} \Pi^c &= \left(\frac{1}{2} \pi^{\alpha\beta} s_\alpha \wedge s_\beta \right)^c = \left(\frac{1}{2} \pi^{\alpha\beta} \right)^c (s_\alpha \wedge s_\beta)^v + \left(\frac{1}{2} \pi^{\alpha\beta} \right)^v (s_\alpha \wedge s_\beta)^c = \\ &= \frac{1}{2} \pi^{\alpha\beta} s_\alpha^v \wedge s_\beta^v + \frac{1}{2} \pi^{\alpha\beta} (s_\alpha^c \wedge s_\beta^v + s_\alpha^v \wedge s_\beta^c) = \frac{1}{2} \pi^{\alpha\beta} \mathcal{V}_\alpha \wedge \mathcal{V}_\beta + \\ &+ \frac{1}{2} \pi^{\alpha\beta} \left((\mathcal{X}_\alpha - L_{\alpha\gamma}^\delta y^\gamma \mathcal{V}_\delta) \wedge \mathcal{V}_\beta + \mathcal{V}_\alpha \wedge (\mathcal{X}_\beta - L_{\beta\gamma}^\delta y^\gamma \mathcal{V}_\delta) \right) = \\ &= \pi^{\alpha\beta} \mathcal{X}_\alpha \wedge \mathcal{V}_\beta + \left(\frac{1}{2} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} - \pi^{\delta\beta} L_{\delta\gamma}^\alpha \right) y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta. \end{aligned}$$

□

Proposition 1.7.7 *The complete lift Π^c is a Poisson bivector on $\mathcal{T}E$.*

Proof. Using the relation (1.7.9) by straightforward computation we obtain $[\Pi^c, \Pi^c] = 0$ which ends the proof. □

Proposition 1.7.8 *The Poisson structure Π^c has the following property*

$$\Pi^c = -\mathcal{L}_C \Pi^c,$$

which means that $(\mathcal{T}E, \Pi^c)$ is a homogeneous Poisson manifold.

Proof. A direct computation in local coordinates yields

$$\begin{aligned} \mathcal{L}_C \Pi^c &= \mathcal{L}_{y^\varepsilon \mathcal{V}_\varepsilon} \left(\pi^{\alpha\beta} \mathcal{X}_\alpha \wedge \mathcal{V}_\beta + \left(\frac{1}{2} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} - \pi^{\delta\beta} L_{\delta\gamma}^\alpha \right) y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta \right) \\ &= \mathcal{L}_{y^\varepsilon \mathcal{V}_\varepsilon} (\pi^{\alpha\beta} \mathcal{X}_\alpha) \wedge \mathcal{V}_\beta - \pi^{\alpha\beta} \mathcal{X}_\alpha \wedge \mathcal{V}_\beta + \mathcal{L}_{y^\varepsilon \mathcal{V}_\varepsilon} \left(\frac{1}{2} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} y^\gamma \mathcal{V}_\alpha \right) \wedge \mathcal{V}_\beta \\ &\quad - \frac{1}{2} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta - (\mathcal{L}_{y^\varepsilon \mathcal{V}_\varepsilon} \pi^{\delta\beta} L_{\delta\gamma}^\alpha y^\gamma \mathcal{V}_\alpha) \wedge \mathcal{V}_\beta + \pi^{\delta\beta} L_{\delta\gamma}^\alpha y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta \\ &= -\pi^{\alpha\beta} \mathcal{X}_\alpha \wedge \mathcal{V}_\beta - \frac{1}{2} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta + \pi^{\delta\beta} L_{\delta\gamma}^\alpha y^\gamma \mathcal{V}_\alpha \wedge \mathcal{V}_\beta \\ &= -\Pi^c \end{aligned}$$

□

Definition 1.7.7 *Let us consider a Poisson bivector on E given by (1.7.1) then the horizontal lift of Π to $\mathcal{T}E$ is the bivector defined by*

$$\Pi^H = \frac{1}{2}\pi^{\alpha\beta}(x)\delta_\alpha \wedge \delta_\beta.$$

Proposition 1.7.9 *The horizontal lift Π^H is a Poisson bivector if and only if Π is a Poisson bivector on E and*

$$\mathcal{R}\left((\pi^\# \rho)^h, (\pi^\# \theta)^h\right) = 0, \quad \forall \rho, \theta \in \Gamma(E). \quad (1.7.10)$$

Proof. The Poisson condition $[\Pi, \Pi] = 0$ leads to the relation (1.7.2) and $[\Pi^H, \Pi^H] = 0$ yields

$$\sum_{(\varepsilon, \delta, \alpha)} \left(\pi^{\alpha\beta} \pi^{\gamma\delta} L_{\beta\gamma}^\varepsilon + \pi^{\alpha\beta} \sigma_\beta^i \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} \right) \delta_\varepsilon \wedge \delta_\alpha \wedge \delta_\delta + \pi^{\alpha\beta} \pi^{\gamma\delta} \mathcal{R}_{\beta\gamma}^\varepsilon \mathcal{V}_\varepsilon \wedge \delta_\alpha \wedge \delta_\gamma = 0,$$

and $\pi^{\alpha\beta} \pi^{\gamma\delta} \mathcal{R}_{\beta\gamma}^\varepsilon = 0$, which is the local expression of the relation (1.7.10). \square

We recall that two Poisson structures are compatible if the bivectors ω_1 and ω_2 satisfy the condition

$$[\omega_1, \omega_2] = 0$$

By straightforward computation in local coordinates we get:

Proposition 1.7.10 *The Poisson bivector Π^H is compatible with the complete lift Π^c if and only if the following relations hold*

$$\begin{aligned} & \pi^{r\beta} \pi^{\alpha s} \left(\frac{\partial \mathcal{N}_r^\gamma}{\partial y^s} - \frac{\partial \mathcal{N}_s^\gamma}{\partial y^r} \right) - \pi^{r\gamma} \pi^{s\alpha} L_{sr}^\beta = 0, \\ & \pi^{rs} \left(\delta_r(a^{\alpha\beta}) - a^{l\alpha} \frac{\partial N_r^\beta}{\partial y^l} + a^{l\beta} \frac{\partial N_r^\alpha}{\partial y^l} - \right. \\ & \quad \left. - \pi^{\theta\beta} R_{r\theta}^\alpha + (\pi^{\varepsilon\beta} L_{\varepsilon\gamma}^\theta - \pi^{\varepsilon\theta} L_{\varepsilon\gamma}^\beta) y^\gamma \frac{\partial N_r^\alpha}{\partial y^\theta} + \right. \\ & \quad \left. + \sigma_r^i \frac{\partial \pi^{\varepsilon\beta}}{\partial x^i} y^\gamma L_{\varepsilon\gamma}^\alpha + \pi^{\varepsilon\beta} L_{\varepsilon\gamma}^\alpha N_r^\gamma - \pi^{\varepsilon\beta} \sigma_r^i \frac{\partial L_{\varepsilon\gamma}^\alpha}{\partial x^i} y^\gamma \right) = 0. \end{aligned}$$

where we have denoted

$$a^{\alpha\beta} = \sigma_\varepsilon^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} y^\varepsilon + N_\varepsilon^\alpha \pi^{\varepsilon\beta} - N_\varepsilon^\beta \pi^{\varepsilon\alpha}.$$

See [84] for the particular case of tangent bundle.

1.7.3 Canonical Poisson structure

On Lie algebroid $(\mathcal{T}E^*, [\cdot, \cdot]_{\mathcal{T}E^*}, \sigma^1)$ we have the canonical symplectic section ω_E given by (1.5.6) which induces a vector bundle isomorphism

$$\flat_{\omega_E} : E^* \rightarrow E, \quad i_\zeta \omega_E \in E^* \rightarrow \zeta \in E.$$

Definition 1.7.8 *The canonical Poisson bivector is given by*

$$\Lambda = \flat_{\omega_E} \omega_E.$$

It follows that

$$\Lambda(dF, dG) = -\omega_E(\flat(dF), \flat(dG)), \quad F, G \in C^\infty(E^*)$$

and in local coordinates we get

$$\Lambda = \mathcal{P}^\alpha \wedge \mathcal{X}_\alpha + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{P}^\beta \wedge \mathcal{P}^\gamma.$$

Remark 1.7.3 *The Schouten-Nijenhuis bracket $[\Lambda, \Lambda]$ leads, locally, to the expression*

$$\frac{1}{3} \sum_{(\alpha, \beta, \gamma)} (\sigma_\alpha^i \frac{\partial L_{\beta\gamma}^\varepsilon}{\partial x^i} + L_{\alpha\delta}^\varepsilon L_{\beta\gamma}^\delta) \mu_\varepsilon \mathcal{P}^\beta \wedge \mathcal{P}^\alpha \wedge \mathcal{P}^\gamma$$

and $[\Lambda, \Lambda] = 0$ follows from the structure equations on Lie algebroids (1.2.9).

Definition 1.7.9 *Let us consider a Poisson bivector on E given by*

$$\Pi = \frac{1}{2} \pi^{\alpha\beta}(x) s_\alpha \wedge s_\beta$$

then the horizontal lift of Π to $\mathcal{T}E^*$ is the bivector defined by

$$\Pi^H = \frac{1}{2} \pi^{\alpha\beta}(x) \delta_\alpha^* \wedge \delta_\beta^*.$$

Proposition 1.7.11 *The horizontal lift Π^H is a Poisson bivector if and only if Π is a Poisson bivector on E and*

$$\mathcal{R} \left((\pi^\# \rho)^h, (\pi^\# \theta)^h \right) = 0, \quad \forall \rho, \theta \in \Gamma(E^*) \quad (1.7.11)$$

Proof. The Poisson condition $[W, W] = 0$ leads to the relation

$$\sum_{(\alpha, \varepsilon, \delta)} (\pi^{\alpha\beta} \pi^{\gamma\delta} L_{\beta\gamma}^\varepsilon + \pi^{\alpha\beta} \sigma_\beta^i \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i}) = 0$$

and $[\Pi^H, \Pi^H] = 0$ yields

$$\sum_{(\varepsilon, \delta, \alpha)} \left(\pi^{\alpha\beta} \pi^{\gamma\delta} L_{\beta\gamma}^\varepsilon + \pi^{\alpha\beta} \sigma_\beta^i \frac{\partial \pi^{\varepsilon\delta}}{\partial x^i} \right) \delta_\varepsilon^* \wedge \delta_\alpha^* \wedge \delta_\delta^* + \pi^{\alpha\beta} \pi^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} \mathcal{P}^\varepsilon \wedge \delta_\alpha^* \wedge \delta_\gamma^* = 0$$

and it results $\pi^{\alpha\beta} \pi^{\gamma\delta} \mathcal{R}_{\beta\gamma\varepsilon} = 0$, which is the local expression of the condition (1.7.11). \square

Proposition 1.7.12 *If the connection \mathcal{N} on $\mathcal{T}E^*$ is defined by a linear connection ∇ with the coefficients $\Gamma_{\alpha\beta}^\gamma$ on the Lie algebroid E , the bivector Π^H has the following form*

$$\Pi^H = \frac{1}{2} \pi^{\alpha\beta} \mathcal{Q}_\alpha \wedge \mathcal{Q}_\beta + \frac{1}{2} \pi^{\alpha\beta} \mu_\gamma \mu_\theta \Gamma_{\alpha\varepsilon}^\gamma \Gamma_{\beta\delta}^\theta \mathcal{P}^\varepsilon \wedge \mathcal{P}^\delta + \pi^{\alpha\beta} \mu_\gamma \Gamma_{\beta\varepsilon}^\gamma \mathcal{Q}_\alpha \wedge \mathcal{P}^\varepsilon. \quad (1.7.12)$$

Proof. The coefficients of the nonlinear connection have the form $N_{\alpha\beta} = \mu_\gamma \Gamma_{\alpha\beta}^\gamma$ and introducing the relation $\delta_\alpha^* = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\beta} \mathcal{P}^\beta$ into the expression of Π^H it results (1.7.12).

Proposition 1.7.13 *If \mathcal{N} is a symmetric nonlinear connection then the canonical Poisson bivector has the form*

$$\Lambda = \mathcal{P}^\alpha \wedge \delta_\alpha^*.$$

Proof. We have

$$\begin{aligned} \Lambda &= \mathcal{P}^\alpha \wedge \mathcal{Q}_\alpha + \frac{1}{2} \mu_\alpha L_{\beta\gamma}^\alpha \mathcal{P}^\beta \wedge \mathcal{P}^\gamma = \mathcal{P}^\alpha \wedge (\delta_\alpha^* - \mathcal{N}_{\alpha\beta} \mathcal{P}^\beta) + \frac{1}{2} \mu_\gamma L_{\beta\alpha}^\gamma \mathcal{P}^\beta \wedge \mathcal{P}^\alpha \\ &= \mathcal{P}^\alpha \wedge \delta_\alpha^* - \frac{1}{2} \mathcal{N}_{\alpha\beta} \mathcal{P}^\alpha \wedge \mathcal{P}^\beta - \frac{1}{2} \left(\mathcal{N}_{\alpha\beta} + \mu_\gamma L_{\beta\alpha}^\gamma \right) \mathcal{P}^\alpha \wedge \mathcal{P}^\beta \\ &= \mathcal{P}^\alpha \wedge \delta_\alpha^* - \frac{1}{2} \mathcal{N}_{\alpha\beta} \mathcal{P}^\alpha \wedge \mathcal{P}^\beta - \frac{1}{2} \mathcal{N}_{\beta\alpha} \mathcal{P}^\alpha \wedge \mathcal{P}^\beta \\ &= \mathcal{P}^\alpha \wedge \delta_\alpha^* - \frac{1}{2} \mathcal{N}_{\alpha\beta} \mathcal{P}^\alpha \wedge \mathcal{P}^\beta - \frac{1}{2} \mathcal{N}_{\alpha\beta} \mathcal{P}^\beta \wedge \mathcal{P}^\alpha \\ &= \mathcal{P}^\alpha \wedge \delta_\alpha^* - \frac{1}{2} \mathcal{N}_{\alpha\beta} \mathcal{P}^\alpha \wedge \mathcal{P}^\beta + \frac{1}{2} \mathcal{N}_{\alpha\beta} \mathcal{P}^\alpha \wedge \mathcal{P}^\beta = \mathcal{P}^\alpha \wedge \delta_\alpha^*. \quad \square \end{aligned}$$

Proposition 1.7.14 *If Π^H is a Poisson bivector and \mathcal{N} is a symmetric nonlinear connection, then Π^H is compatible with the canonical Poisson structure Λ if and only if the following relations fulfilled*

$$\sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} + \pi^{\varepsilon\alpha} \left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\beta} - L_{\varepsilon\gamma}^\beta \right) - \pi^{\varepsilon\beta} \left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\alpha} - L_{\varepsilon\gamma}^\alpha \right) = 0, \quad (1.7.13)$$

$$\pi^{\alpha\beta} \mathcal{R}_{\alpha\gamma\varepsilon} = 0. \quad (1.7.14)$$

Proof. If \mathcal{N} is symmetric then $\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L_{\alpha\beta}^\gamma$ and with respect with the basis $\{\delta_\alpha^*, \mathcal{P}^\alpha\}$ it results $\Lambda = \mathcal{P}^\alpha \wedge \delta_\alpha^*$. By a straightforward computation we obtain

$$\begin{aligned} [\Pi^H, \Lambda] &= -\frac{1}{2} \left(\sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} + \pi^{\varepsilon\alpha} \left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\beta} - L_{\varepsilon\gamma}^\beta \right) \right) \delta_\alpha^* \wedge \delta_\beta^* \wedge \mathcal{P}^\gamma \\ &\quad + \frac{\pi^{\varepsilon\beta}}{2} \left(\frac{\partial \mathcal{N}_{\varepsilon\gamma}}{\partial \mu_\alpha} - L_{\varepsilon\gamma}^\alpha \right) \delta_\alpha^* \wedge \delta_\beta^* \wedge \mathcal{P}^\gamma \\ &\quad + \pi^{\alpha\beta} \mathcal{R}_{\alpha\gamma\varepsilon} \mathcal{P}^\varepsilon \wedge \delta_\beta^* \wedge \mathcal{P}^\gamma. \end{aligned}$$

and $[\Pi^H, \Lambda] = 0$ is equivalent with the relations (1.7.13), (1.7.14). \square

Remark 1.7.4 *If the nonlinear connection \mathcal{N} is defined by a linear connection ∇ with the coefficients $\Gamma_{\alpha\beta}^\gamma$ on the Lie algebroid E then we obtain the conditions*

$$\begin{aligned} \sigma_\gamma^i \frac{\partial \pi^{\alpha\beta}}{\partial x^i} + \pi^{\varepsilon\alpha} \left(\Gamma_{\varepsilon\gamma}^\beta - L_{\varepsilon\gamma}^\beta \right) - \pi^{\varepsilon\beta} \left(\Gamma_{\varepsilon\gamma}^\alpha - L_{\varepsilon\gamma}^\alpha \right) &= 0, \\ \pi^{\alpha\beta} \mu_\varepsilon \left(\sigma_\alpha^i \frac{\partial \Gamma_{\beta\gamma}^\varepsilon}{\partial q^i} - \sigma_\beta^i \frac{\partial \Gamma_{\alpha\gamma}^\varepsilon}{\partial q^i} + \Gamma_{\alpha\theta}^\varepsilon \Gamma_{\beta\gamma}^\theta - \Gamma_{\beta\theta}^\varepsilon \Gamma_{\alpha\gamma}^\theta - L_{\alpha\beta}^\theta \Gamma_{\theta\gamma}^\varepsilon \right) &= 0. \end{aligned}$$

2 Optimal Control

The Lie geometric methods in control theory have been applied by many authors (see for instance. [18, 56, 15, 71]). One of the most important issues in the geometric approach is the analysis of the solution to the optimal control problem as provided by Pontryagin's Maximum Principle; that is, the curve $c(t) = (x(t), u(t))$ is an optimal trajectory if there exists a lifting of $x(t)$ to the dual space $(x(t), p(t))$ satisfying the Hamilton equations, together with a maximization condition for the Hamiltonian with respect to the control variables $u(t)$. The purpose of this part is to study the drift less control affine systems (distributional systems) with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid in the case of constant rank of distribution. Author's papers [49, 96, 98, 101, 102, 106, 108] are used in writing this part.

We prove that the framework of Lie algebroids is better than cotangent bundle in order to solve some problems of drift less control affine systems. In the first chapter the known results on the optimal control systems are recalled by geometric viewpoint. In the next chapter the distributional systems are presented and the relation between the Hamiltonians on E^* and T^*M is given. We investigate the cases of holonomic and nonholonomic distributions with constant rank.

In the holonomic case, we will consider the Lie algebroid being just the distribution whereas in the nonholonomic case (i.e., strong bracket generating distribution) the Lie algebroid is the tangent bundle with the basis given by vectors of distribution completed by the first Lie brackets. In the both cases illustrative examples are presented. Also, the case of distribution D with non-constant rank is studied and some interesting examples are given. In the last chapter we present the intrinsic relation between the distributional systems and sub-Riemannian geometry. Thus, the optimal trajectory of our distributional systems are the geodesics in the framework of sub-Riemannian geometry. We investigate two classical cases: Grusin plan and Heisenberg group [49], equipped with positive homogeneous costs (Randers metric [7]).

2.1 Geometric viewpoint of the optimal control

Let M be a smooth n -dimensional manifold. We consider the control system

$$\frac{dx^i}{dt} = f^i(x, u),$$

where $x \in M$ and the control u takes values in an open subset Ω of R^m . Let x_0 and x_1 be two points of M . An optimal control problem consists of finding the trajectories of our control system which connects x_0 and x_1 and minimizing the cost

$$\min \int_0^T L(x(t), u(t)) dt, \quad x(0) = x_0, \quad x(T) = x_1,$$

where L is the *Lagrangian* or *running cost*.

Necessary conditions for a trajectory to be an extreme are given by Pontryagin Maximum Principle. The Hamiltonian reads as

$$H(x, p, u) = \langle p, f(x, u) \rangle - L(x, u), \quad p \in T^*M,$$

while the maximization condition with respect to the control variables u , namely

$$H(x(t), p(t), u(t)) = \max_v H(x(t), p(t), v),$$

leads to

$$\frac{\partial H}{\partial u} = 0.$$

The extreme trajectories satisfy the Hamilton equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}. \quad (2.1.1)$$

From a geometric viewpoint the pair (x^i, u^a) , where $i = \overline{1, n}$ and $a = \overline{1, m}$, can be understood as a local coordinate pair of a manifold E , that is fibered over M by the projection $\pi : E \rightarrow M$. The functions $f^i(x, u)$ are the components of a vector field $X = f^i(x, u) \frac{\partial}{\partial x^i}$ along π , that is, of a fibered mapping $X : E \rightarrow TM$ from the bundle (E, π, M) to the tangent bundle (TM, τ, M) such that $\tau \circ X = \pi$. The admissible curves of the control system are the curves $\gamma : I \subset R \rightarrow E$ such that

$$X(\gamma(t)) = \frac{d}{dt}(\pi(\gamma(t))).$$

The optimal control problem consists of obtaining the admissible curves that minimize the cost

$$\min \int_I L(\gamma(t))dt, \quad L \in C^\infty(E),$$

and satisfy certain boundary conditions not to be considered here.

The Hamiltonian H is a real-valued function defined on the fibered product $T^*M \times_M E$ that is given by

$$H(\mu, v) = \langle \mu, X(v) \rangle - L(v),$$

for any $(\mu, v) \in T^*M \times_M E$.

The critical equations follow from asking a vector field X_H defined along a map $pr_1 : T^*M \times_M E \rightarrow T^*M$ to satisfy the symplectic equations

$$i_{X_H}\omega = dH,$$

where $\omega = dx^i \wedge dp_i$. Since

$$dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial u^a} du^a,$$

we obtain that the solution of the above equations is the vector field

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i},$$

defined on the subset

$$\frac{\partial H}{\partial u^a} = 0$$

of $T^*M \times_M E$, and therefore, the critical trajectories are the integral curves of the above vector field, namely

$$\frac{\partial H}{\partial u^a} = 0, \quad \dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \quad (2.1.2)$$

By a control system on the Lie algebroid (see [71]) $\pi : E \rightarrow M$ with the control space $\tau : A \rightarrow M$ we mean a section ρ of E along τ . A trajectory of the system ρ is an integral curve of the vector field $\sigma(\rho)$. Given the cost function $\mathcal{L} \in C^\infty(A)$, we have to minimize the integral of \mathcal{L} over the set of those system trajectories which satisfy certain boundary conditions. The Hamiltonian function $\mathcal{H} \in C^\infty(E^* \times_M A)$ is defined by

$$\mathcal{H}(\mu, u) = \langle \mu, \rho(u) \rangle - \mathcal{L}(u),$$

whereas the associated Hamiltonian control system ρ_H is given by the symplectic equation

$$i_{\rho_H}\omega_E = d^E \mathcal{H}.$$

In local coordinates, the solution of the previous equation reads as

$$\rho_H = \frac{\partial \mathcal{H}}{\partial \mu_\alpha} \mathcal{X}_\alpha - \left(\sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial q^i} + \mu_\gamma L_{\alpha\beta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\beta} \right) \mathcal{P}^\alpha,$$

on the subset where

$$\frac{\partial \mathcal{H}}{\partial u^A} = 0.$$

Therefore, the critical trajectories are given by

$$\frac{\partial \mathcal{H}}{\partial u^A} = 0, \quad \frac{dq^i}{dt} = \sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial \mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\sigma_\alpha^i \frac{\partial \mathcal{H}}{\partial q^i} - \mu_\gamma L_{\alpha\beta}^\gamma \frac{\partial \mathcal{H}}{\partial \mu_\beta}. \quad (2.1.3)$$

2.2 Distributional systems

Let M be a smooth n -dimensional manifold. We consider the distributional system (drift less control-affine system)

$$\dot{x} = \sum_{i=1}^m u_i X_i(x), \quad (2.2.1)$$

where $x \in M$, X_1, X_2, \dots, X_m are smooth vector fields on M and the control $u = (u_1, u_2, \dots, u_m)$ takes values in an open subset Ω of R^m . The vector fields $X_i, i = \overline{1, m}$, generate a distribution $D \subset TM$ such that the rank of D is constant. Let x_0 and x_1 be two points of M . An optimal control problem consists of finding those trajectories of the distributional system which connect x_0 and x_1 , while minimizing the cost

$$\min_{u(\cdot)} \int_I \mathcal{F}(u(t)) dt, \quad (2.2.2)$$

where \mathcal{F} is a Minkowski norm (positive homogeneous) on D .

Remark 2.2.1 *We can associate to any positive homogeneous cost \mathcal{F} on the Lie algebroid E a cost F on $Im\sigma \subset TM$ defined by*

$$F(v) = \{\mathcal{F}(u) \mid u \in E_x, \quad \sigma(u) = v\}, \quad (2.2.3)$$

where $v \in (Im\sigma)_x \subset T_x M$, $x \in M$.

A piecewise smooth curve $c : I \subset R \rightarrow M$ is called horizontal if the tangent vectors are in D , i.e. $\dot{c}(t) \in D_{c(t)} \subset TM$ for almost every $t \in I$. Let $u : I \rightarrow E$ be an admissible curve projected by π onto the horizontal curve $c : I \rightarrow M$. The length of the horizontal curve c is defined by

$$length(c) = \int_I \mathcal{F}(u(t)) dt = \int_I F(\dot{c}(t)) dt,$$

and the distance is given by $d(a, b) = \inf length(c)$ where the infimum is taken over all the horizontal curves connecting a and b . The distance is infinite if there is no admissible curve that connects these two points.

Remark 2.2.2 *The energy of a horizontal curve is*

$$E(c) = \frac{1}{2} \int_I F^2(\dot{c}(t)) dt,$$

and it can easily be proved that if a curve is parameterized to a constant speed, then it minimizes the length integral if and only if it minimizes the energy integral.

For the 2-homogeneous Lagrangian $L = \frac{1}{2}F^2$ and $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$ we have

$$\mathcal{L} = L \circ \sigma.$$

Further, L on TM is a Lagrangian with constraints. According to the Pontryagin Maximum Principle, the Hamiltonian is recast as

$$H(x, p, u) = \langle p, \dot{x} \rangle - L(x, u).$$

If the equations

$$\frac{\partial H(x, p, u)}{\partial u} = 0,$$

permit us to find in a unique way u as a smooth function of (x, p) then we can write the Hamiltonian system without any dependence on the control. This nice situation happens always for distributional systems with quadratic cost

$$\min \int_I \sum_{i=1}^m u_i^2 dt.$$

If the cost is not quadratic, then we cannot guarantee that the Hamiltonian can be calculated without dependence on the control. However, there exist several situations when the Hamiltonian can still be found.

Proposition 2.2.1 *The relation between the Hamiltonian H on cotangent bundle T^*M and the Hamiltonian \mathcal{H} on dual bundle E^* is given by*

$$H(p) = \mathcal{H}(\mu), \quad \mu = \sigma^*(p), \quad p \in T_x^*M, \quad \mu \in E_x^*. \quad (2.2.4)$$

Proof. The Fenchel-Legendre dual of Lagrangian L is the Hamiltonian H given by

$$\begin{aligned} H(p) &= \sup_v \{ \langle p, v \rangle - L(v) \} = \sup_v \{ \langle p, v \rangle - \mathcal{L}(u); \sigma(u) = v \} \\ &= \sup_u \{ \langle p, \sigma(u) \rangle - \mathcal{L}(u) \} = \sup_u \{ \langle \sigma^*(p), u \rangle - \mathcal{L}(u) \} = \mathcal{H}(\sigma^*(p)), \end{aligned}$$

and we get

$$H(p) = \mathcal{H}(\mu), \mu = \sigma^*(p), p \in T_x^*M, \mu \in E_x^*,$$

or locally

$$\mu_\alpha = \sigma_\alpha^i p_i, \quad (2.2.5)$$

where the Hamiltonian $H(p)$ is degenerate on $\text{Ker} \sigma^* \subset T^*M$. \square

2.2.1 Holonomic distribution

We assume for the beginning that the distribution $D = \langle X_1, X_2, \dots, X_m \rangle$ is holonomic, which means that $[X_i, X_j] \in D$ for every $i, j = \overline{1, m}$, $i \neq j$. In order to apply the theory of Lie algebroids we consider $E = D$ with the inclusion as anchor $\sigma : E \rightarrow TM$. From the Frobenius theorem, the distribution D is integrable, it determines a foliation on M and two points can be joined if and only if they are situated on the same leaf.

We consider the following distributional system with positive homogeneous cost [98]:

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3, \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x^1 \\ x^2 \\ 1 \end{pmatrix},$$

and

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t)) dt, \quad \mathcal{F}(u) = \sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1, \quad 0 \leq \varepsilon < 1.$$

We are looking for the trajectories starting from the point $(1, 1, 0)^t$ and parameterized by arclength. The associated distribution $D = \langle X_1, X_2 \rangle$ is holonomic, because

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3},$$

and therefore $[X_1, X_2] = X_1$. In the case of the Lie algebroid, we consider $E = \langle X_1, X_2 \rangle$ and the anchor $\sigma : E \rightarrow T\mathbb{R}^3$ has the components

$$\sigma_\alpha^i = \begin{pmatrix} 1 & x^1 \\ 0 & x^2 \\ 0 & 1 \end{pmatrix}, \quad (2.2.6)$$

and we get the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\sqrt{(u_1)^2 + (u_2)^2} + \varepsilon u_1 \right)^2.$$

Using [48],[83] we can find the Hamiltonian on E^* given by

$$\mathcal{H}(\mu) = \frac{1}{2} \left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} - \frac{\varepsilon \mu_1}{1-\varepsilon^2} \right)^2. \quad (2.2.7)$$

Remark 2.2.3 Using (2.2.4) we can calculate the Hamiltonian H on T^*M given by $H(x, p) = \mathcal{H}(\mu)$, $\mu = \sigma^*(p)$, where

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x^1 & x^2 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

We get that

$$H(x, p) = \frac{1}{2} \left(\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{(p_1 x^1 + p_2 x^2 + p_3)^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1}{1-\varepsilon^2} \right)^2. \quad (2.2.8)$$

Unfortunately, with $H(x, p)$ from (2.2.8) the Hamilton equations on T^*M lead to a complicated system of implicit differential equations.

We will use the geometric model of a Lie algebroid. From the relation $[X_\alpha, X_\beta] = L_{\alpha\beta}^\gamma X_\gamma$ we obtain the non-zero components $L_{12}^1 = 1$, $L_{21}^1 = -1$ while from (2.1.3) we deduce that

$$\dot{x}^1 = \frac{\partial \mathcal{H}}{\partial \mu_1} + x^1 \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{x}^2 = x^2 \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{x}^3 = \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad (2.2.9)$$

$$\dot{\mu}_1 = -\mu_1 \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{\mu}_2 = \mu_1 \frac{\partial \mathcal{H}}{\partial \mu_1},$$

where

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mu_1} &= \frac{(1+\varepsilon^2)\mu_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon \sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}{1-\varepsilon^2} - \frac{\varepsilon \mu_1^2}{(1-\varepsilon^2)^3 \sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}, \\ \frac{\partial \mathcal{H}}{\partial \mu_2} &= \frac{\mu_2}{1-\varepsilon^2} - \frac{\varepsilon \mu_1 \mu_2}{(1-\varepsilon^2)^2 \sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}}}. \end{aligned} \quad (2.2.10)$$

The form of the last relations leads to the following change of variables

$$\mu_1(t) = (1-\varepsilon^2)r(t) \operatorname{sech} \theta(t),$$

$$\mu_2(t) = \sqrt{1-\varepsilon^2} r(t) \tanh \theta(t).$$

In these circumstances we have

$$\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} = |r|,$$

whereas

$$\dot{\mu}_1 = -\mu_1 \frac{\partial \mathcal{H}}{\partial \mu_2},$$

yields

$$\sqrt{1 - \varepsilon^2} \left(\frac{\dot{r}}{r} - \dot{\theta} \tanh \theta \right) = r(-\tanh \theta + \varepsilon \sec h \theta \tanh \theta). \quad (2.2.11)$$

From

$$\dot{\mu}_2 = \mu_1 \frac{\partial \mathcal{H}}{\partial \mu_1},$$

we get

$$\sqrt{1 - \varepsilon^2} \left(\frac{\dot{r}}{r} \tanh \theta + \dot{\theta} \operatorname{sech}^2 \theta \right) = r((1 + \varepsilon)^2 \operatorname{sech}^2 \theta - \varepsilon \operatorname{sech} \theta - \varepsilon \operatorname{sech}^3 \theta). \quad (2.2.12)$$

Now, reducing $\dot{\theta}$ and $\frac{\dot{r}}{r}$ from the relations (2.2.11) and (2.2.12), we obtain

$$\sqrt{1 - \varepsilon^2} \dot{r} = r^2 \varepsilon \operatorname{sech} \theta \tanh \theta (\varepsilon \operatorname{sech} \theta - 1),$$

and

$$\sqrt{1 - \varepsilon^2} \dot{\theta} = r(\varepsilon \operatorname{sech} \theta - 1)^2. \quad (2.2.13)$$

The last two relations lead to

$$\frac{\dot{r}}{\dot{\theta}} = \frac{r \varepsilon \operatorname{sech} \theta \tanh \theta}{\varepsilon \operatorname{sech} \theta - 1},$$

and respectively to

$$\frac{1}{r} dr = \frac{\varepsilon \operatorname{sech} \theta \tanh \theta}{\varepsilon \operatorname{sech} \theta - 1} d\theta,$$

with the solution

$$\ln |r| = -\ln(\varepsilon \sec h \theta - 1) - \ln c.$$

Therefore

$$|r| = \frac{1}{c(\varepsilon \operatorname{sech} \theta - 1)}.$$

Since the geodesics are parameterized by arclength, the conclusion corresponds exactly to the 1/2 level of the Hamiltonian and so we have

$$\mathcal{H} = \frac{r^2}{2} (1 - \varepsilon \operatorname{sech} \theta)^2 = \frac{1}{2c^2}.$$

Now, $c = \pm 1$ and

$$r = \pm \frac{1}{\varepsilon \operatorname{sech} \theta - 1}.$$

From the relation (2.2.13) we have

$$\frac{d\theta}{dt} = \frac{\sqrt{1 - \varepsilon^2}}{1 - \varepsilon \operatorname{sech} \theta},$$

and respectively

$$t = \sqrt{1 - \varepsilon^2} \int \frac{1}{1 - \varepsilon \operatorname{sech} \theta} d\theta,$$

The relation

$$\dot{\mu}_1 = -\mu_1 \dot{x}^3,$$

implies that

$$x^3(\theta) = \ln \frac{c_1(1 - \varepsilon \operatorname{sech} \theta)}{(1 - \varepsilon^2) \operatorname{sech} \theta}, \quad c_1 \in \mathbb{R}.$$

Since we are looking for the trajectories starting from the point $(1, 1, 0)^t$, we have

$$\ln \frac{c_1}{1 + \varepsilon} = 0 \Rightarrow c_1 = 1 + \varepsilon,$$

and so

$$x^3(\theta) = \ln \frac{1 - \varepsilon \operatorname{sech} \theta}{(1 - \varepsilon) \operatorname{sech} \theta} = \ln \frac{\cosh \theta - \varepsilon}{1 - \varepsilon}.$$

The relation

$$\frac{\dot{x}^2}{x^2} = -\frac{\dot{\mu}_1}{\mu_1},$$

leads to

$$x^2(\theta) = \frac{c_2(1 - \varepsilon \operatorname{sech} \theta)}{(1 - \varepsilon^2) \operatorname{sech} \theta},$$

whereas from $x^2(0) = 1$ we get $c_2 = 1 + \varepsilon$. These lead to

$$x^2(\theta) = \frac{\cosh \theta - \varepsilon}{1 - \varepsilon}.$$

We obtain also that

$$\dot{\mu}_2 = \mu_1 \left(\dot{x}^1 - x^1 \frac{\partial \mathcal{H}}{\partial \mu_2} \right) = \mu_1 \dot{x}^1 + x^1 \dot{\mu}_1,$$

and, consequently, $\mu_2 = \mu_1 x^1 + c_3$. Further,

$$x^1(\theta) = \frac{\sinh \theta}{\sqrt{1 - \varepsilon^2}} \pm \frac{c_3(1 - \varepsilon \operatorname{sech} \theta)}{(1 - \varepsilon^2) \operatorname{sech} \theta}.$$

From $x^1(0) = 1$ we obtain that $c_3 = 1 + \varepsilon$ and this yields

$$x^1(\theta) = \frac{\sinh \theta}{\sqrt{1 - \varepsilon^2}} + \frac{\cosh \theta - \varepsilon}{1 - \varepsilon}.$$

Remark 2.2.4 If $\varepsilon = 0$ we regain the case of distributional systems with quadratic cost with the solution

$$x^1(t) = \sinh t + \cosh t, \quad x^2(t) = \cosh t, \quad x^3(t) = \ln \cosh t.$$

2.2.2 Nonholonomic distribution

We assume that the distribution $D = \langle X_1, X_2, \dots, X_m \rangle$ is nonholonomic and is strong bracket generating [118], i.e. sections of D and first iterated brackets span the entire tangent space TM . By a well-known theorem of Chow, the system is controllable, that is any two points are connected through a horizontal curve (M is assumed to be connected). We also suppose that the vectors $\mathcal{B} = \{X_1, X_2, \dots, X_m, [X_i, X_j]\}$ determine a base in TM . The space $E = TM$ with the base \mathcal{B} is a Lie algebroid over M with at least one structural function nonzero. The anchor $\sigma : E \rightarrow TM$ is just the identity and the matrix corresponding to σ is determined by the base vectors.

The control system can be written as

$$\dot{x} = \sum_{i=1}^m u_i X_i(x) + 0X_{m+1} + \dots + 0X_n,$$

with

$$\min_{u(\cdot)} \int_0^T \mathcal{L}(u(t)) dt.$$

To solve this minimization problem we consider the Lagrangian

$$\tilde{\mathcal{L}} = \mathcal{L} + \sum_{k=m+1}^n \lambda_k u^k,$$

(λ_k are the Lagrange multipliers) and still work via the maximum principle but at the level of Lie algebroids. We set $\mu_i = \frac{\partial \tilde{\mathcal{L}}}{\partial u^i}$ and

$$\mathcal{H} = \sum_{i=1}^n \mu_i u^i - \tilde{\mathcal{L}},$$

thus obtaining $\mu_j = \frac{\partial \mathcal{L}}{\partial u^j}$, $\mu_k = \lambda_k$ with $j = \overline{1, m}$ and $k = \overline{m+1, n}$. Since \mathcal{L} is 2-homogeneous with respect to u^i , $i = \overline{1, m}$, we get

$$\mathcal{H} = \sum_{i=1}^m \frac{\partial \mathcal{L}}{\partial u^i} u^i - \mathcal{L} = \mathcal{L},$$

with the constraints $\mu_k = \lambda_k$, $k = \overline{m+1, n}$.

We consider the following distributional system with positive homogeneous cost [98]:

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in R^3, \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x^1 \end{pmatrix},$$

and

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t))dt, \quad \mathcal{F}(u) = \sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1, \quad 0 \leq \varepsilon < 1.$$

We are looking for the trajectories starting from the origin and parameterized by arclength. We have

$$X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3},$$

$$X_3 = [X_1, X_2] = \frac{\partial}{\partial x^3},$$

and $\langle X_1, X_2, X_3 \rangle \equiv TR^3$, hence the distribution $D = \langle X_1, X_2 \rangle$ of constant rank is strong bracket generating.

Remark 2.2.5 *We can work, as in classical case, directly on the cotangent bundle by computing the Hamiltonian $H(x, p) = \mathcal{H}(x, \mu)$, $\mu = \sigma^*(p)$. Since*

$$\mathcal{H} = \frac{1}{2} \left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{1-\varepsilon^2}} - \frac{\varepsilon \mu_1}{1-\varepsilon^2} \right)^2, \quad \mu_3 = \lambda,$$

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x^1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

we obtain

$$H = \frac{1}{2} \left(\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{(p_2 + p_3 x^1)^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1}{1-\varepsilon^2} \right)^2.$$

Unfortunately, from the Hamilton equations on T^*M a very complicated system of implicit differential equations is obtained.

We will use a different approach. Let us take $M = R^3$ and $E = TM$ with the basis $\{X_1, X_2, X_3\}$. E is a Lie algebroid over M with at least one structural function nonzero. The anchor $\sigma : E \rightarrow TM$ is just the identity and the matrix of σ with respect to the basis of E and TM basis is

$$\sigma_\alpha^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x^1 & 1 \end{pmatrix}.$$

From the structure equations of Lie algebroids and the relation $[X_\alpha, X_\beta] = L_{\alpha\beta}^\gamma X_\gamma$ we obtain the non-zero structural functions $L_{12}^3 = 1$, $L_{21}^3 = -1$. Now,

using the Hamilton equations on Lie algebroids (2.1.3) we get the following systems of differential equations

$$\dot{x}^1 = \frac{\partial \mathcal{H}}{\partial \mu_1}, \quad \dot{x}^2 = \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad \dot{x}^3 = x^1 \frac{\partial \mathcal{H}}{\partial \mu_2}, \quad (2.2.14)$$

and

$$\begin{cases} \dot{\mu}_1 = -L_{12}^3 \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} = -\lambda \frac{\partial \mathcal{H}}{\partial \mu_2}, \\ \dot{\mu}_2 = -L_{21}^3 \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} = \lambda \frac{\partial \mathcal{H}}{\partial \mu_1}, \\ \dot{\mu}_3 = 0 \Rightarrow \lambda = ct, \end{cases} \quad (2.2.15)$$

where

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mu_1} &= \frac{(1 + \varepsilon^2) \mu_1}{(1 - \varepsilon^2)^2} - \frac{\varepsilon \sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}}}{1 - \varepsilon^2} - \frac{\varepsilon \mu_1^2}{(1 - \varepsilon^2)^3 \sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}}}, \\ \frac{\partial \mathcal{H}}{\partial \mu_2} &= \frac{\mu_2}{1 - \varepsilon^2} - \frac{\varepsilon \mu_1 \mu_2}{(1 - \varepsilon^2)^2 \sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}}}. \end{aligned}$$

We may use to the following transformations

$$\mu_1(t) = (1 - \varepsilon^2) r(t) (a \cos A\theta(t) - b \sin A\theta(t)), \quad (2.2.16)$$

$$\mu_2(t) = \sqrt{1 - \varepsilon^2} r(t) (a \sin A\theta(t) + b \cos A\theta(t)), \quad (2.2.16')$$

with $a^2 + b^2 = 1$. We also have $\sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}} = |r|$. Further, (2.2.15) yields

$$\begin{aligned} c_1 \left(\frac{\dot{r}}{r} (a \sin A\theta + b \cos A\theta) + A\dot{\theta} (a \cos A\theta - b \sin A\theta) \right) &= \\ = (1 + \varepsilon^2) (a \cos A\theta - b \sin A\theta) - \varepsilon (1 + (a \cos A\theta - b \sin A\theta)^2) \end{aligned} \quad (2.2.17)$$

and

$$\begin{aligned} c_1 \left(\frac{\dot{r}}{r} (a \cos A\theta - b \sin A\theta) - A\dot{\theta} (a \sin A\theta + b \cos A\theta) \right) &= \\ = -(a \sin A\theta + b \cos A\theta) + \varepsilon (a \cos A\theta - b \sin A\theta) (a \sin A\theta + b \cos A\theta), \end{aligned} \quad (2.2.18)$$

where $c_1 = \frac{(1 - \varepsilon^2) \sqrt{1 - \varepsilon^2}}{\lambda}$. Reducing $\dot{\theta}$ and $\frac{\dot{r}}{r}$ from (2.2.17) and (2.2.18), we get

$$c_1 \frac{\dot{r}}{r} = \varepsilon (a \sin A\theta + b \cos A\theta) (\varepsilon a \cos A\theta - \varepsilon b \sin A\theta - 1) \quad (2.2.19)$$

and

$$c_1 A \dot{\theta} = (1 - \varepsilon(a \cos A\theta - b \sin A\theta))^2. \quad (2.2.20)$$

These lead to

$$t = \frac{(1 - \varepsilon^2)\sqrt{1 - \varepsilon^2}A}{\lambda} \int \frac{d\theta}{(1 - \varepsilon(a \cos A\theta - b \sin A\theta))^2}.$$

From (2.2.19) and (2.2.20) we obtain

$$\frac{dr}{r} = \frac{A\varepsilon(a \sin A\theta + b \cos A\theta)}{\varepsilon(a \cos A\theta - b \sin A\theta) - 1} d\theta$$

and

$$r = \frac{1}{c(1 - \varepsilon(a \cos A\theta - b \sin A\theta))}.$$

Since the geodesics are parameterized by arclength this corresponds exactly to the $1/2$ level of the Hamiltonian and we have

$$\mathcal{H} = \frac{r^2}{2} (1 - \varepsilon(a \cos A\theta - b \sin A\theta))^2 = \frac{1}{2c^2}.$$

So, $c = \pm 1$ and

$$r = \pm \frac{1}{1 - \varepsilon(a \cos A\theta - b \sin A\theta)}.$$

From (2.2.16) we obtain

$$\begin{aligned} \mu_1(t) &= \pm \frac{(1 - \varepsilon^2)(a \cos A\theta - b \sin A\theta)}{1 - \varepsilon(a \cos A\theta - b \sin A\theta)} \\ \mu_2(t) &= \frac{\sqrt{1 - \varepsilon^2}(a \sin A\theta + b \cos A\theta)}{1 - \varepsilon(a \cos A\theta - b \sin A\theta)}. \end{aligned}$$

Since $\dot{\mu}_2 = \lambda \dot{x}^1$, we also have $x^1(\theta) = \frac{\mu_2}{\lambda} - a_1$. As we are looking for geodesics with start from the origin, we have $a_1 = \frac{\sqrt{1 - \varepsilon^2}b}{\lambda(1 - \varepsilon a)}$ and therefore

$$x^1(\theta) = \frac{\sqrt{1 - \varepsilon^2}(a \sin A\theta + b \cos A\theta)}{\lambda(1 - \varepsilon(a \cos A\theta - b \sin A\theta))} - \frac{\sqrt{1 - \varepsilon^2}b}{\lambda(1 - \varepsilon a)}.$$

From $\dot{\mu}_1 = -\lambda \dot{x}^2$ we get

$$x^2(\theta) = \frac{(1 - \varepsilon^2)(b \sin A\theta - a \cos A\theta)}{\lambda(1 - \varepsilon(a \cos A\theta - b \sin A\theta))} + \frac{(1 - \varepsilon^2)a}{\lambda(1 - \varepsilon a)}.$$

Finally, because $\dot{x}^3 = x^1 \dot{x}^2$ a straightforward computation leads to

$$\begin{aligned} x^3(\theta) &= \frac{(1 - \varepsilon^2)\sqrt{1 - \varepsilon^2}A}{\lambda^2} \int \frac{(a \sin A\theta + b \cos A\theta)^2}{(1 - \varepsilon(a \cos A\theta - b \sin A\theta))^3} d\theta - \\ &\quad - \frac{(1 - \varepsilon^2)Ab}{\lambda^2} \int \frac{a \sin A\theta + b \cos A\theta}{(1 - \varepsilon(a \cos A\theta - b \sin A\theta))^2} d\theta \end{aligned}$$

Remark 2.2.6 For $\varepsilon = 0$ we obtain the distributional systems with quadratic cost with the solution .

$$\begin{aligned}
x^1(t) &= \frac{a \sin \lambda t - b(1 - \cos \lambda t)}{\lambda}, \quad a^2 + b^2 = 1. \\
x^2(t) &= \frac{b \sin \lambda t + a(1 - \cos \lambda t)}{\lambda}, \\
x^3(t) &= \frac{t}{2\lambda} + \frac{b^2 - a^2}{4\lambda^2} \sin 2\lambda t - \frac{ab}{\lambda^2} \cos^2 \lambda t + \frac{ab}{\lambda^2} \cos \lambda t - \frac{b^2}{a^2} \sin \lambda t.
\end{aligned}$$

2.2.3 Distributional systems with no constant rank of distribution

Let us consider in the three dimensional space R^3 the drift less control affine system [102]

$$\dot{x}(t) = u^1 X_1 + u^2 X_2 + u^3 X_3, \quad (2.2.21)$$

with

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix},$$

and minimizing the cost

$$\min_{u(\cdot)} \int F(u(t)) dt, \quad (2.2.22)$$

where $F = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2} + \varepsilon u^1$, $0 \leq \varepsilon < 1$ is the positive homogeneous cost (Randers metric). The distribution D is generated by the vectors X_1, X_2, X_3 and we can write $D = \{X_1, X_2, X_3\}$. We observe that

$$\text{rank} D = \begin{cases} 3 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

In the canonical base $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ of R^3 we have $X_1 = \frac{\partial}{\partial x}$, $X_2 = x \frac{\partial}{\partial y}$, $X_3 = x \frac{\partial}{\partial z}$ and the Lie brackets are given by

$$[X_1, X_2] = \frac{\partial}{\partial y} = X_4 \notin D, \quad [X_1, X_3] = \frac{\partial}{\partial z} = X_5 \notin D, \quad [X_2, X_3] = 0.$$

It results that the distribution is nonholonomic, but is bracket generating, because the vector fields $\{X_1, X_2, X_3, X_4 = [X_1, X_2], X_5 = [X_1, X_3]\}$ generate the entire space R^3 .

From (2.2.21) we obtain

$$\frac{dx}{dt} = u^1 = s^1, \quad \frac{dy}{dt} = u^2 x = s^2, \quad \frac{dz}{dt} = u^3 x = s^3.$$

The cost function can be written in the form ($x \neq 0$)

$$\begin{aligned} F &= \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2} + \varepsilon u^1 = \sqrt{(s^1)^2 + \frac{(s^2)^2}{x^2} + \frac{(s^3)^2}{x^2}} + \varepsilon s^1 \\ &= \sqrt{g_{ij} s^i s^j} + \sum_{i=1}^3 b^i s^i \end{aligned}$$

(Einstein's summation) where $b^1 = \varepsilon$, $b^2 = 0$, $b^3 = 0$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/x^2 & 0 \\ 0 & 0 & 1/x^2 \end{pmatrix}.$$

The Lagrangian has the form $L = \frac{1}{2}F^2$ and using [83] (Th. 4.5 pp. 191) we obtain the Hamiltonian in the form

$$H = \frac{1}{2} \left(\sqrt{\tilde{g}_{ij} p_i p_j} - \tilde{b}^i p_i \right), \quad (2.2.23)$$

where

$$\tilde{g}_{ij} = \frac{1}{1-b^2} g^{ij} + \frac{1}{(1-b^2)^2} b^i b^j, \quad \tilde{b}^i = \frac{1}{1-b^2} b^i, \quad b = \sqrt{g_{ij} b^i b^j},$$

and g^{ij} is the inverse of the matrix g_{ij} . In these conditions we obtain that

$$b^2 = \varepsilon^2, \quad \tilde{b}^1 = \frac{\varepsilon}{1-\varepsilon^2}, \quad \tilde{b}^2 = 0, \quad \tilde{b}^3 = 0,$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 \end{pmatrix}$$

and and it results

$$\tilde{g}^{ij} = \begin{pmatrix} \frac{1}{(1-\varepsilon^2)^2} & 0 & 0 \\ 0 & \frac{x^2}{1-\varepsilon^2} & 0 \\ 0 & 0 & \frac{x^2}{1-\varepsilon^2} \end{pmatrix}$$

From (2.2.23) we obtain

$$H = \frac{1}{2} \left(\sqrt{\frac{p_1^2}{(1-\varepsilon^2)^2} + \frac{p_2^2 x^2}{1-\varepsilon^2} + \frac{p_3^2 x^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1}{1-\varepsilon^2} \right)^2 \quad (2.2.24)$$

or, in the equivalent form

$$H = \frac{(1+\varepsilon^2) p_1^2}{2(1-\varepsilon^2)^2} + \frac{(p_2^2 + p_3^2) x^2}{2(1-\varepsilon^2)} - \frac{\varepsilon p_1}{1-\varepsilon^2} \sqrt{\frac{p_1^2}{(1-\varepsilon^2)^2} + \frac{(p_2^2 + p_3^2) x^2}{1-\varepsilon^2}}.$$

In the case $x = 0$ we obtain

$$L = \frac{1}{2} F^2 = \frac{(1+\varepsilon)^2 u_1^2}{2},$$

with the constraints $\dot{y} = 0$, $\dot{z} = 0$. Using Lagrange multipliers we obtain

$$L_1 = L + \lambda_1 \dot{y} + \lambda_2 \dot{z},$$

and from Legendre transformation it results

$$H_1 = \frac{1}{2} \frac{p_1^2}{(1 + \varepsilon)^2}.$$

For $x = 0$ from (2.2.24) we have

$$H = \frac{1}{2} \left(\frac{p_1}{1 - \varepsilon^2} - \frac{\varepsilon p_1}{1 - \varepsilon^2} \right)^2 = \frac{1}{2} \frac{p_1^2}{(1 + \varepsilon)^2},$$

which leads to the equality

$$H|_{x=0} = H_1.$$

Next, if we denote

$$\Theta = \frac{p_1^2}{(1 - \varepsilon^2)^2} + \frac{(p_2^2 + p_3^2) x^2}{1 - \varepsilon^2},$$

then the Hamilton's equations (2.1.1) lead to the following differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_1} = \frac{(1 + \varepsilon^2)p_1}{(1 - \varepsilon^2)^2} - \frac{\varepsilon}{1 - \varepsilon^2} \sqrt{\Theta} - \frac{\varepsilon p_1^2}{(1 - \varepsilon^2)^3} \frac{1}{\sqrt{\Theta}}, \quad (2.2.25)$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial p_2} = \frac{p_2 x^2}{1 - \varepsilon^2} - \frac{\varepsilon p_1 p_2 x^2}{(1 - \varepsilon^2)^2} \frac{1}{\sqrt{\Theta}}, \quad (2.2.26)$$

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_3} = \frac{p_3 x^2}{1 - \varepsilon^2} - \frac{\varepsilon p_1 p_3 x^2}{(1 - \varepsilon^2)^2} \frac{1}{\sqrt{\Theta}}, \quad (2.2.27)$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = -\frac{(p_2^2 + p_3^2) x}{1 - \varepsilon^2} + \frac{\varepsilon p_1 (p_2^2 + p_3^2) x}{(1 - \varepsilon^2)^2} \frac{1}{\sqrt{\Theta}}, \quad (2.2.28)$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2 = a = \text{const.}$$

$$\frac{dp_3}{dt} = -\frac{\partial H}{\partial z} = 0 \Rightarrow p_3 = b = \text{const.}$$

In these conditions the relation $\Theta = \frac{p_1^2}{(1 - \varepsilon^2)^2} + \frac{(p_2^2 + p_3^2) x^2}{1 - \varepsilon^2}$ leads to the following change of variables:

$$x(t) = \frac{\sqrt{1 - \varepsilon^2} r(t) \sin A\theta(t)}{\sqrt{a^2 + b^2}}, \quad p_1(t) = (1 - \varepsilon^2) r(t) \cos A\theta(t).$$

It results $\Theta = r^2(t)$ and from (2.2.25) we get

$$\frac{dx}{dt} = \frac{(1 + \varepsilon^2) r \cos A\theta}{1 - \varepsilon^2} - \frac{\varepsilon r}{1 - \varepsilon^2} - \frac{\varepsilon r \cos^2 A\theta}{1 - \varepsilon^2}.$$

But

$$\frac{dx}{dt} = \frac{\sqrt{1-\varepsilon^2}}{\sqrt{a^2+b^2}} \left(\dot{r} \sin A\theta + rA \dot{\theta} \cos A\theta \right),$$

and it results

$$c_1 \left(\dot{r} \sin A\theta + rA \dot{\theta} \cos A\theta \right) = (1+\varepsilon^2)r \cos A\theta - \varepsilon r(1+\cos^2 A\theta), \quad (2.2.29)$$

where we have denoted

$$c_1 = \frac{(1-\varepsilon^2)\sqrt{1-\varepsilon^2}}{\sqrt{a^2+b^2}}.$$

The equation (2.2.28) yields

$$\frac{dp_1}{dt} = \frac{\sqrt{a^2+b^2}}{\sqrt{1-\varepsilon^2}} (-r \sin A\theta + \varepsilon r \cos A\theta \sin A\theta).$$

But

$$\frac{dp_1}{dt} = (1-\varepsilon^2) \left(\dot{r} \cos A\theta - rA \dot{\theta} \sin A\theta \right),$$

which leads to

$$c_1 \left(\dot{r} \cos A\theta - rA \dot{\theta} \sin A\theta \right) = -r \sin A\theta + \varepsilon r \cos A\theta \sin A\theta. \quad (2.2.30)$$

The equation (2.2.29) multiplied by $\cos A\theta$, minus equation (2.2.30) multiplied by $\sin A\theta$ lead to the equation

$$c_1 A \frac{d\theta}{dt} = (1-\varepsilon \cos A\theta)^2, \quad (2.2.31)$$

and it results

$$t = \frac{A(1-\varepsilon^2)\sqrt{1-\varepsilon^2}}{\sqrt{a^2+b^2}} \int \frac{1}{(1-\varepsilon \cos A\theta)^2} d\theta$$

Moreover, the equation (2.2.29) multiplied by $\sin A\theta$, plus equation (2.2.30) multiplied by $\cos A\theta$ lead to the equation

$$c_1 \frac{dr}{dt} = \varepsilon r \sin A\theta (\varepsilon \cos A\theta - 1). \quad (2.2.32)$$

From the equations (2.2.31) and (2.2.32) it results

$$\frac{dr}{r} = -A \frac{\varepsilon \sin A\theta}{1-\varepsilon \cos A\theta} d\theta,$$

which leads to the following result

$$\ln r = -\ln c(1-\varepsilon \cos A\theta), \quad c \in \mathbb{R},$$

and we get

$$r(t) = \frac{1}{c(1 - \varepsilon \cos A\theta(t))}. \quad (2.2.33)$$

Using (2.2.33) the Hamiltonian become

$$\begin{aligned} H &= \frac{1}{2}(1 + \varepsilon^2)r^2 \cos^2 A\theta + \frac{1}{2}r^2 \sin^2 A\theta - \varepsilon r^2 \cos A\theta \\ &= \frac{1}{2}r^2 + \frac{1}{2}\varepsilon^2 r^2 \cos^2 A\theta - \varepsilon r^2 \cos A\theta \\ &= \frac{r^2}{2} (1 - \varepsilon \cos A\theta) \\ &= \frac{1}{2c^2} \end{aligned}$$

Considering the integral curves parameterized by arclength, that corresponds to fix the level $\frac{1}{2}$ of the Hamiltonian, we have $c = \pm 1$ and it results

$$r = \pm \frac{1}{1 - \varepsilon \cos A\theta}. \quad (2.2.34)$$

In these conditions, from (2.2.34) we obtain

$$x(t) = \pm \frac{\sqrt{1 - \varepsilon^2}}{\sqrt{a^2 + b^2}} \frac{\sin A\theta(t)}{1 - \varepsilon \cos A\theta(t)}. \quad (2.2.35)$$

The differential equation (2.2.26) yields

$$\frac{dy}{dt} = \frac{ar^2 \sin^2 A\theta}{a^2 + b^2} (1 - \varepsilon \cos A\theta) = \frac{a}{a^2 + b^2} \frac{\sin^2 A\theta}{1 - \varepsilon \cos A\theta}. \quad (2.2.36)$$

From (2.2.31) and (2.2.36) it results

$$y(t) = \frac{aA(1 - \varepsilon^2)^{3/2}}{(a^2 + b^2)^{3/2}} \int \frac{\sin^2 A\theta}{(1 - \varepsilon \cos A\theta)^3} d\theta(t). \quad (2.2.37)$$

In the same way we obtain

$$z(t) = \frac{bA(1 - \varepsilon^2)^{3/2}}{(a^2 + b^2)^{3/2}} \int \frac{\sin^2 A\theta}{(1 - \varepsilon \cos A\theta)^3} d\theta(t). \quad (2.2.38)$$

In the particular case of $\varepsilon = 0$ we obtain a distributional system with quadratic cost

$$F = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2},$$

with the solution

$$x(t) = \pm \frac{\sin \alpha t}{\alpha},$$

$$y(t) = \frac{at}{2\alpha^2} - \frac{a \sin 2\alpha t}{4\alpha^{3/2}},$$

$$z(t) = \frac{bt}{2\alpha^2} - \frac{a \sin 2\alpha t}{4\alpha^{3/2}}.$$

where $\alpha = \sqrt{a^2 + b^2}$, which are the geodesics in the framework of the so called sub-Riemannian geometry [14].

2.2.4 Distributional systems with degenerate cost

Let us consider in the two dimensional space R^2 the drift less control affine system [106]

$$\dot{x}(t) = u^1 X_1 + u^2 X_2, \quad (2.2.39)$$

with

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x \end{pmatrix},$$

and minimizing the cost

$$\min_{u(\cdot)} \int_I F(u(t)) dt, \quad (2.2.40)$$

where $F = \sqrt{(u^1)^2 + (u^2)^2} + u^1$ is the positive homogeneous cost (Kropina metric).

I have to remark that in the case $u^1 \leq 0$, $u^2 = 0$ it results $F = 0$ that is we obtain a degenerate cost (metric). The distribution D is generated by the vectors X_1, X_2 and we have

$$\text{rank} D = \begin{cases} 2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

In the canonical base of R^2 given by $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$ we can write

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x \frac{\partial}{\partial y}$$

and the Lie brackets are given by

$$[X_1, X_2] = \frac{\partial}{\partial y} = X_3 \notin D.$$

It results that the distribution is nonholonomic, but is bracket generating, because the vectors fields $\{X_1, X_2, X_3 = [X_1, X_2]\}$ generate the entire space R^2 . From (2.2.39) we obtain

$$\begin{aligned} \frac{dx}{dt} &= u^1 = s^1. \\ \frac{dy}{dt} &= u^2 x = s^2. \end{aligned}$$

The cost function can be written in the following form ($x \neq 0$)

$$F = \sqrt{(u^1)^2 + (u^2)^2} + u^1 = \sqrt{(s^1)^2 + \frac{(s^2)^2}{x^2}} + s^1 = \sqrt{g_{ij} s^i s^j} + \sum_{i=1}^3 b^i s^i$$

(Einstein's summation, $i, j = 1, 2$) where $b^1 = 1$, $b^2 = 0$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 1 & 1/x^2 \end{pmatrix}.$$

The Lagrangian function has the form $L = \frac{1}{2}F^2$ and using [83] we obtain the Hamiltonian in the form

$$H = \frac{1}{2} \left(\frac{g^{ij} p_i p_j}{2b^i p_i} \right)^2, \quad (2.2.41)$$

where

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 1 & x^2 \end{pmatrix},$$

is the inverse of the matrix g_{ij} . In these conditions we obtain that

$$H = \frac{1}{2} \left(\frac{p_1^2 + x^2 p_2^2}{2p_1} \right)^2,$$

or, in the equivalent form

$$H = \frac{p_1^2}{8} + \frac{p_2^2}{8p_1^2} x^4 + \frac{p_2^2}{4} x^2.$$

The Hamilton's equations (2.1.1) lead to the following differential equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_1} = \frac{p_1}{4} - \frac{p_2^4 x^4}{4p_1^3} \quad (2.2.42)$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial p_2} = \frac{p_2^3 x^4}{2p_1^2} + \frac{p_2 x^2}{2} \quad (2.2.43)$$

$$\frac{dp_1}{dt} = -\frac{\partial H}{\partial x} = -\frac{p_2^4 x^3}{2p_1^2} - \frac{p_2^2 x}{2} \quad (2.2.44)$$

$$\frac{dp_2}{dt} = -\frac{\partial H}{\partial y} = 0 \Rightarrow p_2 = a = \text{const.}$$

and it results

$$\begin{aligned} \frac{dx}{dt} &= \frac{p_1^4 - a^4 x^4}{4p_1^3}, \\ \frac{dy}{dt} &= \frac{a^3 x^4 + ap_1^2 x^2}{2p_1^2}, \\ \frac{dp_1}{dt} &= -\frac{a^2 x}{2} - \frac{a^4 x^3}{2p_1^2}. \end{aligned}$$

In these conditions the expression of the Hamiltonian leads to the following change of variables:

$$x(t) = \frac{r(t) \sin A\theta(t)}{a}, \quad p_1(t) = r(t) \cos A\theta(t) \quad (2.2.45)$$

We obtain the equations

$$\frac{dx}{dt} = \frac{r (\cos^4 A\theta - \sin^4 A\theta)}{4 \cos^3 A\theta},$$

and

$$\frac{dp_1}{dt} = -\frac{ar \sin A\theta}{2} - \frac{ar \sin^3 A\theta}{2 \cos^2 A\theta}.$$

But on the other hand

$$\begin{aligned} \frac{dx}{dt} &= \frac{\dot{r}}{a} \sin A\theta + \frac{r A \dot{\theta}}{a} \cos A\theta \\ \frac{dp_1}{dt} &= \dot{r} \cos A\theta - r A \dot{\theta} \sin A\theta \end{aligned}$$

and it follows

$$\frac{1}{a} \left(\frac{\dot{r}}{r} \sin A\theta + A \dot{\theta} \cos A\theta \right) = \frac{\cos^4 A\theta - \sin^4 A\theta}{4 \cos^3 A\theta}, \quad (2.2.46)$$

$$\frac{1}{a} \left(\frac{\dot{r}}{r} \cos A\theta - A \dot{\theta} \sin A\theta \right) = -\frac{\sin A\theta}{2} - \frac{\sin^3 A\theta}{2 \cos^2 A\theta}. \quad (2.2.47)$$

The equation (2.2.46) multiplied by $\cos A\theta$, minus equation (2.2.47) multiplied by $\sin A\theta$ leads to the equation

$$\frac{A}{a} \frac{d\theta}{dt} = \frac{1}{4 \cos^3 A\theta}, \quad (2.2.48)$$

which yields

$$t = \frac{4A}{a} \int \cos^2 A\theta d\theta,$$

and it follows

$$t = \frac{1}{a} (\sin 2A\theta + 2A\theta).$$

Moreover, the equation (2.2.46) multiplied by $\sin A\theta$, plus equation (2.2.47) multiplied by $\cos A\theta$ leads to the equation

$$\frac{1}{a} \frac{\dot{r}}{r} = -\frac{\sin A\theta}{4 \cos^3 A\theta}. \quad (2.2.49)$$

Using the equations (2.2.48) and (2.2.49) we obtain

$$\frac{dr}{r} = -A \frac{\sin A\theta}{\cos A\theta} d\theta,$$

which leads to the following result

$$\ln |r| + \ln c_1 = \ln |\cos A\theta|, \quad c_1 \in R^+,$$

and we get

$$r(t) = \frac{1}{c_1} \cos A\theta(t). \quad (2.2.50)$$

Using the change of variables (2.2.45) the Hamiltonian become

$$H = \frac{r^2 \cos^2 A\theta}{8} + \frac{r^2 \sin^4 A\theta}{8 \cos^2 A\theta} + \frac{r^2 \sin^2 A\theta}{4} = \frac{r^2}{8 \cos^2 A\theta},$$

and from (2.2.50) we get

$$H = \frac{1}{8c_1^2}.$$

Considering the integral curves parameterized by arclength, that corresponds to fix the level 1/2 of the Hamiltonian, we have $c_1 = \pm 1/2$ and it results

$$r = \pm 2 \cos A\theta. \quad (2.2.51)$$

which together with (2.2.45) lead to the result

$$x(t) = \pm \frac{\sin 2A\theta(t)}{a} \quad (2.2.52)$$

The differential equation (2.2.43) yields by direct computation to

$$\frac{dy}{dt} = \frac{2 \sin^2 A\theta}{a}, \quad (2.2.53)$$

and from (2.2.48) and (2.2.53) it follows

$$dy = \frac{2A}{a^2} \sin^2(2A\theta) d\theta,$$

which yields

$$y(t) = \frac{A\theta(t)}{a^2} - \frac{\sin 4A\theta(t)}{4a^2}. \quad (2.2.54)$$

Finally, the solution is

$$\begin{aligned} x(t) &= \pm \frac{\sin 2A\theta(t)}{a}, \\ y(t) &= \frac{A\theta(t)}{a^2} - \frac{\sin 4A\theta(t)}{4a^2}. \end{aligned}$$

2.3 Sub-Riemannian geometry

If M is a smooth n -dimensional manifold then a sub-Riemannian structure on M is a pair (D, g) , where D is a distribution of rank m and g is a Riemannian metric on D . A sub-Riemannian manifold (M, D, g) is a smooth manifold M equipped with a sub-Riemannian structure [118, 85, 14].

A piecewise smooth curve $c : I \subset \mathbb{R} \rightarrow M$ is called *horizontal* if its tangent vectors are in D , i.e. $\dot{c}(t) \in D_{c(t)} \subset TM$, for almost every $t \in I$. In sub-Riemannian geometry the *length* of a horizontal curve c is defined by

$$L(c) = \int_I \sqrt{g(\dot{c}(t))} dt, \quad (2.3.1)$$

where g is a Riemannian metric on D . The *distance* from a to b is

$$d(a, b) = \inf(L(c)),$$

where the infimum is taken over all horizontal curves connecting a to b . The distance is assumed to be infinite if there is no horizontal curve that connects these two points.

If locally the distribution D of rank m is generated by X_i , $i = \overline{1, m}$, a sub-Riemannian structure on M is locally given by a control system

$$\dot{x} = \sum_{i=1}^m u_i(t) X_i(x), \quad (2.3.2)$$

of constant rank m , with the controls $u(\cdot)$. The *controlled paths* are obtained by integrating the above system. If D is assumed to be *bracket generating*, i.e. sections of D and iterated brackets span the entire tangent space TM , by a well-known theorem of Chow [25] the system (2.3.2) is *controllable*, that is for any two points a and b there exists a horizontal curve which connects these points (M is assumed to be connected).

The concept of the sub-Riemannian geometry can be extended to a more general setting, by replacing the Riemannian metric with a Finslerian one. For the theory of optimal control this extension is equivalent to the change of the quadratic cost of a control affine system with a positive homogeneous cost. Also, the case when the rank of D is not constant may produce interesting examples. We do not intend to develop a comprehensive study of the sub-Riemannian geometry. In the present section we introduce two particular sub-Finslerian geometries [49]: the Grushin plane and the Heisenberg group, endowed with some special Randers metrics [7]. The geodesics of these geometries are obtained by using two different approaches: a direct application of the Pontryagin Maximum Principle for the Grushin plane and the same principle but combined with some results on Lie algebroids for the Heisenberg group.

Definition 2.3.1 A sub-Finslerian structure on M is a triple (E, σ, \mathcal{F}) where

- 1) (E, π, M) is a vector bundle over M , with the projection map π .
- 2) $\sigma : E \rightarrow TM$ is a morphism of vector bundles.
- 3) \mathcal{F} is a Finsler metric on E , i.e. $\mathcal{F} : E \rightarrow [0, \infty)$ and satisfies the following properties:
 - a) \mathcal{F} is C^∞ on $E \setminus \{0\}$.
 - b) $\mathcal{F}(\lambda u) = \lambda \mathcal{F}(u)$ for $\lambda > 0$ and $u \in E_x, x \in M$.
 - c) For each $y \in E_x \setminus \{0\}$ the quadratic form

$$g_y(u, v) = \frac{1}{2} \frac{\partial \mathcal{F}^2}{\partial s \partial t} (y + su + tv)_{s,t=0}$$

$u, v \in E_x, x \in M$, is positive definite.

Example 2.3.1 (i) $E = M \times \mathbb{R}^m$, $\{X_1, \dots, X_m\}$ is a system of m vector fields on M , $\sigma : E \rightarrow TM$, given by

$$\sigma(x, u) = \sum_{i=1}^m u_i(t) X_i(x), \quad (2.3.3)$$

and \mathcal{F} is a Minkowski norm on \mathbb{R}^m .

(ii) $E = D$, $\sigma : D \rightarrow TM$ the inclusion and \mathcal{F} a Finsler metric on D .

We can associate to any sub-Finslerian structure (E, σ, \mathcal{F}) a Finsler metric on $Im\sigma \subset TM$, defined as following

$$F(v) = \inf_u \{ \mathcal{F}(u) \mid u \in E_x, \sigma(u) = v \}. \quad (2.3.4)$$

for each

$$v \in (Im\sigma)_x \subset T_x M, x \in M.$$

Definition 2.3.2 A curve $u : I \rightarrow E$ is called admissible if there is an absolute continuous curve $c : I \rightarrow M$, called horizontal, such that $\pi(u(t)) = c(t)$ and $\sigma(u(t)) = \dot{c}(t)$, $t \in I$.

The length of an absolutely continuous horizontal curve $c(t)$, $t \in I$ is

$$length(c) = \int_I \mathcal{F}(u(t)) dt = \int_I F(\dot{c}(t)) dt. \quad (2.3.5)$$

We can also consider the sub-Finslerian distance

$$d(a, b) = \inf length(c),$$

where the infimum is taken over all horizontal curves connecting a and b . This distance is infinite if there is no admissible curve joining a and b . However, if we assume that $Im\sigma$ is bracket generating, Chow's theorem guarantees that the sub-Finslerian distance between points is finite.

Definition 2.3.3 *A length minimizing geodesic or shortly a minimizer is an absolutely continuous horizontal curve on M that makes the distance between two points.*

Remark 2.3.1 *The energy of a horizontal curve is*

$$E(c) = \frac{1}{2} \int_I F^2(\dot{c}(t)) dt,$$

and it can easily be proved that if a curve is parameterized to a constant speed, then it minimize the length integral if and only if it minimize the energy integral.

If we take the Lagrangians $L = \frac{1}{2}F^2$ and $\mathcal{L} = \frac{1}{2}\mathcal{F}^2$ we have $\mathcal{L} = L \circ \sigma$. The Fenchel-Legendre dual of L is the Hamiltonian [49]

$$\begin{aligned} H(p) &= \sup_v \{ \langle p, v \rangle - L(v) \} = \\ &= \sup_v \left\{ \langle p, v \rangle + \sup_u \{ -\mathcal{L}(u); \sigma(u) = v \} \right\} = \\ &= \sup_{u,v} \{ \langle p, v \rangle - \mathcal{L}(u); \sigma(u) = v \} = \\ &= \sup_u \{ \langle p, \sigma(u) \rangle - \mathcal{L}(u) \} = \\ &= \sup_u \{ \langle \sigma^*(p), u \rangle - \mathcal{L}(u) \} = \mathcal{H}(\sigma^*(p)) \end{aligned}$$

Hence

$$H(p) = \mathcal{H}(\mu), \quad \mu = \sigma^*(p) \quad (2.3.6)$$

$p \in T_x^*M$, $\mu \in E_x^*$. The Hamiltonian H on T^*M is degenerate on $\text{Ker} \sigma^*$.

The Hamiltonian H generates a system of differential equations which can be written in terms of canonical coordinates (x, p) in the standard form:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} \quad (2.3.7)$$

Based on the Pontryagin Maximum Principle we can prove:

Theorem 2.3.1 *Let $(x(t), p(t))$ be a solution of the Hamilton equations. Then every sufficient short subarc of $x(t)$ is a minimizing sub-Finslerian geodesic. This subarc is the unique minimizer joining its end points.*

Definition 2.3.4 *The projected curve $x(t)$ will be called normal geodesic or shortly geodesic.*

Remark 2.3.2 *Contrary to Finslerian geometry, not every minimizer is the projection of a solution of (2.3.7) i.e. is a normal geodesic. Those minimizers that are not normal geodesics are called singular geodesics [118, 85].*

2.3.1 Grushin case

We consider the following distributional system with positive homogeneous cost (Grushin Plane) [49]

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^2, \quad X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ x^1 \end{pmatrix}$$

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t)) dt, \quad \mathcal{F}(u) = \|u\| + \langle b, u \rangle, \quad b = (\varepsilon, 0)^t, \quad u = (u^1, u^2)^t, \quad 0 \leq \varepsilon < 1.$$

$$x(0) = 0, \quad x(T) = x_T.$$

We are looking for the geodesics starting from the origin and parameterized by arclength. The distribution $D = \langle X_1, X_2 \rangle$ is bracket generating and has not a constant rank on \mathbb{R}^2 . If we take the regular Lagrangian

$$\mathcal{L} = \frac{1}{2} \mathcal{F}^2 = \frac{1}{2} \left(\sqrt{u_1^2 + u_2^2} + \varepsilon u_1 \right)^2,$$

on $E = D$ and use a result from [83] we obtain a regular Hamiltonian \mathcal{H} on E^*

$$\mathcal{H} = \frac{1}{2} \left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{(1-\varepsilon^2)}} - \frac{\varepsilon \mu_1}{1-\varepsilon^2} \right)^2,$$

and from (2.3.6) we get

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x^1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},$$

and the corresponding Hamiltonian H on T^*M is

$$H = \frac{1}{2} \left(\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{(p_2)^2 (x^1)^2}{(1-\varepsilon^2)}} - \frac{\varepsilon p_1}{1-\varepsilon^2} \right)^2,$$

From the Hamilton equations (2.3.7) we obtain

$$\begin{aligned} \dot{x}^1 &= \frac{(1+\varepsilon)^2 p_1}{(1-\varepsilon^2)^2} - \frac{\varepsilon}{1-\varepsilon^2} \sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{a^2 (x^1)^2}{1-\varepsilon^2}} - \frac{\varepsilon p_1^2}{(1-\varepsilon^2)^3} \frac{1}{\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{a^2 (x^1)^2}{1-\varepsilon^2}}} \\ \dot{x}^2 &= \frac{(x^1)^2 a}{1-\varepsilon^2} - \frac{\varepsilon a (x^1)^2 p_1}{(1-\varepsilon^2)^2} \frac{1}{\sqrt{\frac{(p_1)^2}{(1-\varepsilon^2)^2} + \frac{a^2 (x^1)^2}{1-\varepsilon^2}}} \end{aligned} \quad (2.3.8)$$

$$\dot{p}_1 = -\frac{x^1 a^2}{1 - \varepsilon^2} + \frac{\varepsilon p_1 x^1 a^2}{(1 - \varepsilon^2)^2} \frac{1}{\sqrt{\frac{(p_1)^2}{(1 - \varepsilon^2)^2} + \frac{a^2 (x^1)^2}{1 - \varepsilon^2}}}, \quad p_2 = a = ct.$$

We make the following change of variables

$$\begin{cases} x^1 = \frac{\sqrt{1 - \varepsilon^2}}{a} r(t) \sin A\theta(t), \\ p_1 = (1 - \varepsilon^2) r(t) \cos A\theta(t), \end{cases}$$

and from (2.3.8) we get

$$\begin{aligned} \frac{A(1 - \varepsilon^2)(\sqrt{1 - \varepsilon^2})}{a} \frac{d\theta}{dt} &= (1 - \varepsilon \cos A\theta)^2, \\ \frac{(1 - \varepsilon^2)(\sqrt{1 - \varepsilon^2})}{a} \frac{dr}{dt} &= \varepsilon r \sin A\theta (\varepsilon \cos A\theta - 1). \end{aligned}$$

Hence

$$r = \frac{1}{c(1 - \varepsilon \cos A\theta)}, \quad c \in \mathbb{R}$$

and

$$t = \frac{A(1 - \varepsilon^2)\sqrt{1 - \varepsilon^2}}{a} \int \frac{d\theta}{(1 - \varepsilon \cos A\theta)^2}.$$

But the geodesics are parameterized by arclength, that corresponds to fix the level $1/2$ of the Hamiltonian and we have

$$H = \frac{r^2}{2} (1 - \varepsilon \cos A\theta)^2 = \frac{1}{2c^2},$$

so $c = \pm 1$ and therefore

$$r = \pm \frac{1}{1 - \varepsilon \cos A\theta}$$

and finally we obtain

$$\begin{aligned} x^1 &= \pm \sqrt{1 - \varepsilon^2} \frac{\sin A\theta}{a(1 - \varepsilon \cos A\theta)}, \\ x^2 &= \frac{A(1 - \varepsilon^2)\sqrt{1 - \varepsilon^2}}{a} \int \frac{\sin^2 A\theta}{(1 - \varepsilon \cos A\theta)^3} d\theta. \end{aligned} \tag{2.3.9}$$

Remark 2.3.3 1. The above geodesics are the only minimizers of this sub-Finslerian geometry.

2. For $\varepsilon = 0$ we obtain the geodesics of the Grushin plane endowed with the standard Euclidean metric, i.e. a sub-Riemannian geometry (distributional systems with quadratic cost)

$$x_1 = \pm \frac{\sin at}{a}, \quad x_2 = \frac{t}{2a} - \frac{\sin 2at}{4a^2}. \tag{2.3.10}$$

([14, 37]).

2.3.2 Heisenberg case

We consider the following distributional system with positive homogeneous cost (Heisenberg group) [49]

$$\dot{x} = u^1 X_1 + u^2 X_2, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3, \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{x^2}{2} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{x^1}{2} \end{pmatrix},$$

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t)) dt, \quad \mathcal{F}(u) = \|u\| + \langle b, u \rangle, \quad b = (\varepsilon, 0)^t, \quad u = (u^1, u^2)^t, \quad 0 \leq \varepsilon < 1.$$

We are looking for the geodesics starting from the origin and parameterized by arclength. Here

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - \frac{x^2}{2} \frac{\partial}{\partial x^3}, \\ X_2 &= \frac{\partial}{\partial x^2} + \frac{x^1}{2} \frac{\partial}{\partial x^3}, \\ X_3 &= [X_1, X_2] = \frac{\partial}{\partial x^3} \end{aligned}$$

and $\langle X_1, X_2, X_3 \rangle \equiv T\mathbb{R}^3$, hence the distribution $D = \langle X_1, X_2 \rangle$ of constant rank is strong bracket generating [118].

Remark 2.3.4 *We can try to work directly on the cotangent bundle by computing the Hamiltonian $H(x, p) = \mathcal{H}(x, \mu)$, $\mu = \sigma^*(p)$. Since*

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -x^2/2 \\ 0 & 1 & x^1/2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad (2.3.11)$$

we obtain

$$H = \frac{1}{2} \left(\sqrt{\frac{\left(p_1 - \frac{p_3 x^2}{2}\right)^2}{(1 - \varepsilon^2)^2} + \frac{\left(p_2 + \frac{p_3 x^1}{2}\right)^2}{1 - \varepsilon^2}} - \frac{\varepsilon \left(p_1 - \frac{p_3 x^2}{2}\right)}{1 - \varepsilon^2} \right)^2. \quad (2.3.12)$$

Unfortunately, with this Hamiltonian (2.3.7) is a very complicated system of implicit differential equations.

We will use a different approach. Let us take $M = \mathbb{R}^3$ and $E = TM$ with the basis $\{X_1, X_2, X_3\}$. E is a Lie algebroid over M with at least one

structural function nonzero. The anchor $\sigma : E \rightarrow TM$ is just the identity and the matrix of σ with respect to the basis of E and TM basis is

$$\sigma_\alpha^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x^2}{2} & \frac{x^1}{2} & 1 \end{pmatrix} \quad (2.3.13)$$

Now the above control system can be written

$$\dot{x} = u^1 X_1 + u^2 X_2 + 0 X_3, \quad x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \in \mathbb{R}^3,$$

$$\min_{u(\cdot)} \int_0^T \mathcal{F}(u(t)) dt = \min_{u(\cdot)} \int_0^T \mathcal{L}(u(t)) dt,$$

where $\mathcal{F} = \sqrt{(u^1)^2 + (u^2)^2} + \varepsilon u^1$ and $\mathcal{L} = \frac{1}{2} \mathcal{F}^2$. To solve this problem we form the augmented Lagrangian

$$\tilde{\mathcal{L}} = \mathcal{L}(u) + \lambda u^3,$$

(λ is Lagrange multiplier) and still work via the maximum principle but at the level of the Lie algebroid. If we set

$$\mu_i = \frac{\partial \tilde{\mathcal{L}}}{\partial u^i}, \quad \mathcal{H} = \mu_i u^i - \tilde{\mathcal{L}},$$

we obtain

$$\mu_1 = \frac{\partial \mathcal{L}}{\partial u^1}, \quad \mu_2 = \frac{\partial \mathcal{L}}{\partial u^2}, \quad \mu_3 = \lambda,$$

and therefore

$$\mathcal{H} = \mu_i u^i - \tilde{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial u^1} u^1 + \frac{\partial \mathcal{L}}{\partial u^2} u^2 + \lambda u^3 - \mathcal{L} - \lambda u^3 = \mathcal{L},$$

because \mathcal{L} is 2-homogeneous. Again, using a result from [83], the Hamiltonian on E^* is given by

$$\mathcal{H} = \frac{1}{2} \left(\sqrt{\frac{(\mu_1)^2}{(1-\varepsilon^2)^2} + \frac{(\mu_2)^2}{(1-\varepsilon^2)}} - \frac{\varepsilon \mu_1}{1-\varepsilon^2} \right)^2, \quad \mu_3 = \lambda. \quad (2.3.14)$$

From the structure equations of Lie algebroids (1.2.8) and the relation $[X_\alpha, X_\beta] = L_{\alpha\beta}^\gamma X_\gamma$ we obtain the non-zero structural functions

$$L_{12}^3 = 1, \quad L_{21}^3 = -1.$$

Now, using the Hamilton equations on Lie algebroids (2.1.3) we get the following systems of differential equations

$$\begin{aligned} \dot{x}^1 &= \frac{\partial \mathcal{H}}{\partial \mu_1}, \quad \dot{x}^2 = \frac{\partial \mathcal{H}}{\partial \mu_2}, \\ \dot{x}^3 &= -\frac{x^2}{2} \frac{\partial \mathcal{H}}{\partial \mu_1} + \frac{x^1}{2} \frac{\partial \mathcal{H}}{\partial \mu_2} = \frac{x^1 \dot{x}^2}{2} - \frac{\dot{x}^1 x^2}{2} \end{aligned} \quad (2.3.15)$$

and

$$\begin{aligned} \dot{\mu}_1 &= -L_{12}^3 \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_2} = -\lambda \frac{\partial \mathcal{H}}{\partial \mu_2} \\ \dot{\mu}_2 &= -L_{21}^3 \mu_3 \frac{\partial \mathcal{H}}{\partial \mu_1} = \lambda \frac{\partial \mathcal{H}}{\partial \mu_1} \\ \dot{\mu}_3 &= 0 \Rightarrow \lambda = ct. \end{aligned} \quad (2.3.16)$$

where

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \mu_1} &= \frac{(1 + \varepsilon^2) \mu_1}{(1 - \varepsilon^2)^2} - \frac{\varepsilon \sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}}}{1 - \varepsilon^2} - \frac{\varepsilon \mu_1^2}{(1 - \varepsilon^2)^3 \sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}}} \\ \frac{\partial \mathcal{H}}{\partial \mu_2} &= \frac{\mu_2}{1 - \varepsilon^2} - \frac{\varepsilon \mu_1 \mu_2}{(1 - \varepsilon^2)^2 \sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}}}. \end{aligned}$$

We may use to the following transformations

$$\begin{aligned} \mu_1(t) &= (1 - \varepsilon^2) r(t) (a \cos A\theta(t) - b \sin A\theta(t)) \\ \mu_2(t) &= \sqrt{1 - \varepsilon^2} r(t) (a \sin A\theta(t) + b \cos A\theta(t)) \end{aligned} \quad (2.3.17)$$

such that $a^2 + b^2 = 1$ and we get

$$\sqrt{\frac{(\mu_1)^2}{(1 - \varepsilon^2)^2} + \frac{(\mu_2)^2}{1 - \varepsilon^2}} = |r|.$$

Also, from (2.3.16) we obtain

$$t = \frac{(1 - \varepsilon^2) \sqrt{1 - \varepsilon^2} A}{\lambda} \int \frac{d\theta}{(1 - \varepsilon (a \cos A\theta - b \sin A\theta))^2},$$

and

$$r = \frac{1}{c(1 - \varepsilon (a \cos A\theta - b \sin A\theta))}.$$

But the geodesics are parameterized by arclength, that corresponds to fix the level $1/2$ of the Hamiltonian and we have

$$\mathcal{H} = \frac{r^2}{2} (1 - \varepsilon (a \cos A\theta - b \sin A\theta))^2 = \frac{1}{2c^2}.$$

so $c = \pm 1$ and therefore

$$r = \pm \frac{1}{1 - \varepsilon (a \cos A\theta - b \sin A\theta)}.$$

From (2.3.17) we obtain

$$\begin{aligned}\mu_1(t) &= \pm \frac{(1 - \varepsilon^2)(a \cos A\theta - b \sin A\theta)}{1 - \varepsilon (a \cos A\theta - b \sin A\theta)}, \\ \mu_2(t) &= \frac{\sqrt{1 - \varepsilon^2}(a \sin A\theta + b \cos A\theta)}{1 - \varepsilon (a \cos A\theta - b \sin A\theta)}.\end{aligned}$$

But

$$\dot{\mu}_2 = \lambda \dot{x}^1,$$

so

$$x^1(\theta) = \frac{\mu_2}{\lambda} - a_1.$$

Since we are looking for geodesics starting from the origin, we have $a_1 = \frac{\sqrt{1 - \varepsilon^2}b}{\lambda(1 - \varepsilon a)}$ and therefore

$$x^1(\theta) = \frac{\sqrt{1 - \varepsilon^2}(a \sin A\theta + b \cos A\theta)}{\lambda(1 - \varepsilon (a \cos A\theta - b \sin A\theta))} - \frac{\sqrt{1 - \varepsilon^2}b}{\lambda(1 - \varepsilon a)},$$

and from

$$\dot{\mu}_1 = -\lambda \dot{x}^2,$$

we obtain

$$x^2(\theta) = \frac{(1 - \varepsilon^2)(b \sin A\theta - a \cos A\theta)}{\lambda(1 - \varepsilon (a \cos A\theta - b \sin A\theta))} + \frac{(1 - \varepsilon^2)a}{\lambda(1 - \varepsilon a)}.$$

Finally, from

$$\dot{x}^3 = \frac{x^1 \dot{x}^2}{2} - \frac{\dot{x}^1 x^2}{2},$$

by straightforward computation we get

$$x^3(\theta) = \frac{(1 - \varepsilon^2)\sqrt{1 - \varepsilon^2}A}{2\lambda^2(1 - \varepsilon a)} \int \frac{1 - \cos A\theta}{(1 - \varepsilon (a \cos A\theta - b \sin A\theta))^2} d\theta.$$

Remark 2.3.5 For $\varepsilon = 0$ we obtain the sub-Riemannian case (distributional systems with quadratic cost) with the solutions

$$\begin{aligned}x(t) &= \frac{a \sin \lambda t - b(1 - \cos \lambda t)}{\lambda} \\ y(t) &= \frac{b \sin \lambda t + a(1 - \cos \lambda t)}{\lambda} \\ z(t) &= \frac{\lambda t - \sin \lambda t}{2\lambda^2}, \quad a^2 + b^2 = 1\end{aligned}$$

(see also [14, 37]).

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